Deriving the Real Interest Rate

(see Paul A. Samuelson, *An Exact Consumption-Loan Model with or without the Social Contrivance of Money*, Journal of Political Economy, 1958)

Suppose we have a population that starts at a level of 100 individuals.
Suppose that each year an additional 100 individuals are born.
Suppose that all people live for 3 years.
Thus, after 3 years we have a stable population of 300 individuals.
Suppose people produce 1 unit of chocolate per year, for two years, and zero in their third year.

Production, over four years, would look like the following:

	production				
TIME	total	A	B	C	D
1	100	100			
2	200	100	100		
3	200	0	100	100	
4	0	DOA	0	100	100

Assume chocolates cannot be stored. We consume all production of year X, in year X.
(Samuelson labels this notion the ZERO trade with nature concept)
We know that individuals need chocolate in year 3, to survive.
In their 'middle year' they trade with the 'young', offering output today, for output next year.
Note that the old cannot trade with the young—they have nothing to offer the young and are dead in one year's time.

Trade, over time, would look something like this:

	consumption					
TIME	total	A	B	C	D	E
1	100	100				
2	200	66.6	133.3			
3	200	33.3	33.3	133.3		
4	200	DOA	33.3	33.3	133.3	
5	200		DOA	33.3	33.3	133.3

Note that the naked eye recognizes that the outcome is *socially sub-optimal.*
(We gorge ourselves with chocolate, when we are young, then live subsistence life, in middle age and in old age).

How might we change this picture, in order to improve allocation of chocolates over time? Samuelson tells us we simply need to introduce money into the picture. Money, as a store of value, allows an individual to 'save' notwithstanding the fact that chocolates cannot be stored up and used in a later period. More specifically, money allows the young to trade with the old. The young accept money from the old, knowing full well that the old will not be around to provide chocolates in the future. The key assumption that the young are making? They are wagering that a new generation of young people will appear, and that the money they collect as youngsters can be sent to the new youngsters when these young'uns are old.
Thus the key in this *OVERLAPPING GENERATIONS MODEL* is the assumption of a never ending stream of the unborn.

Let me restate the issue for emphasis. Without money, the middle aged group faces the big squeeze. They 'owe', the oldsters chocolates, and they also need to forgo consuming some of their current chocolate production to the young, in order to secure chocolates for themselves in the next period, when they produce zero. There is no escaping the result: an initial year of gorging followed by two subsistence years.

Smoothed Consumption, over Time, Thanks to Money.

Let's introduce money. We posit that, by fiat, the first generation receives $5 in period 1 and both the first and second generations receive $5 in period 2. Thereafter, no new money enters the system. Suppose further, that the first generation consumes all of his chocolates in both periods, and keeps the $10 he has collected. Finally, assume the second generation consumes all of their chocolates, in their first year, and keeps the $5 they have collected. In period 3, money begins to change hands. And this, we will see, can facilitate a smoothing of consumption over time.

TIME	consumption total	A	B	C	D	E
1	100	100P 100C $5				
2	200	100P 100C $10	100P 100C $5			
3	200	0P 66.6C $0	100P 66.6C $10	100P 66.6C $5		
4	199.94	DOA	0P 66.6C $0	100P 66.6C $10	100P 66.6C $5	
5				0P 66.6C $0	100P 66.6C $10	100P 66.6C $5

In period 3, the young cohort (C) receives $5 from the old, for 33.3 chocolates.
In period 3, the middle cohort (B) also receives $5 from the old, for 33.3 chocolates.
In period 3, the oldsters (A) spend their money, securing 66.6 chocolates in their last year of life.

Note that in period 3, the three cohorts evenly split the total production.

In period 4, note that cohort B, now the old cohort, has $10, and does as A did, sending $5 to both the middle aged and the young for a total of 2/3 of total production.
Note as well that cohort C will end up with $10, endowing them with the wherewithal to secure 66.6 chocolates, one year hence, when they are old.
And so on…

We now have smoothed consumption. The process works because we introduce money and because the young are willing to take as a given the appearance of a new group of youngsters in the not too distant future. Note that the price of chocolates remains constant.
(**Perfect** smoothing of consumption, as we have in this artificial exercise, would result if we assume that consumers have no time preference for chocolate consumption)

Smoothed Consumption, over time, with Climbing Productivity and a Rising Labor Force

What happens to the period-to-period price of chocolates, if we imagine a world with a climbing population and with improving productivity? Let's assume that each cohort is 5% larger than its predecessor. Assume further that consistent advances in schooling lead them to be almost 5% more productive than their predecessor cohort. Output, in this world, grows by 10% per year:

	Production					
TIME	total	A	B	C	D	E
1	100	100				
2	210	100	110			
3	231	0	110	121		
4	254		0	121	133	
5	280			0	133	146
6	307				0	146
7	338					0

Let's now introduce money, in almost the same fashion as in our static output world. The first generation receives $5 in periods 1 and 2. The second generation receives $5.70 in period 2 (obviously, we are establishing initial allocations of money to allow an equilibrium solution to transparently appear almost at once). Thereafter, no new money enters the system. The first generation consumes all of their chocolates in both periods, and keeps the $10 they have collected. The second generation consumes all of their chocolates, in their first year, and keeps the $5.70 they have collected. In period 3, money begins to change hands. Once again, this facilitates a smoothing of consumption over time:

TIME		A	B	C	D	E
1	100	100P 100C $5				
2	210	100P 100C $10	110P 110C $5.70			
3	231	0P 77C $0	110P 77C $10	121P 77C $5.70		
4	254		0P 84.7C $0	121P 84.7C $10	133P 84.7C $5.70	
5	280			121P 93C $0	133P 93C $10	146P 93C $5.70

In period 3:

C_A buys 33 chocolates from C_B for $4.30
C_A buys 44 chocolates from C_C for $5.70
Note: 1 chocolate = 13 cents
In period 4:
C_B buys 36.3 chocolates from C_C for $4.30
C_B buys 48.3 chocolates from C_D for $5.70
Note: 1 chocolate = 12 cents
In period 5:
C_C buys 93 chocolates from C_D and C_E for $10.0
Note: 1 chocolate = 11 cents

Note the interesting result. Prices fall, relative to dollars. This makes perfect sense, as the quantity of goods produced rises continuously, while the stock of money is held constant at $15.70. This is a deflationary world. The decline in prices gives us both the interest rate and the discount rate:
93 chocolates = 36.3 chocolates X $(1+i)^2$ + 44 chocolates X $(1+i)$

Solve for i: i = 10%

93 = (33 X 1.21) + (44 X 1.1)What does this tell us? If we accept the super stylized assumptions, two things:

1. If we introduce money, we balance our consumption, by banking on future generations.
2. The real interest rate equals the growth rate in population and productivity.

What is one breakdown of output growth?

%Δ GDP = %Δ labor force + %Δ labor productivity

In sum, this simple model's suggests that real GDP growth and real interest rates, over long periods, will likely move up and down together.

HOWEVER! The result depends upon successive cohort's embracing the continual arrival of future cohort's, of ever increasing size and productivity. Thus we have a real interest rate tied to expectations of the future real growth rate for productivity and population. Expectations, at the end of the day, are in our heads. Thus the model has multiple, indeed infinite, equilibria.