Maharam-Types and Lyapunov's Theorem for Vector Measures on Banach Spaces*

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Abstract

This paper offers a sufficient condition, based on Maharam (1942) and re-emphasized by Hoover–Keisler (1984), for the validity of Lyapunov's (1940) theorem on the range of an atomless vector measure taking values in an infinite-dimensional Banach space that is not necessarily separable nor has the RNP property. In particular, we obtain an extension of a corresponding result due to Uhl (1969). The proposed condition is also shown to be necessary in the sense formalized in Keisler–Sun (2009), and thereby closes a question of long-standing as regards an infinite-dimensional generalization of the theorem. The result is applied to obtain short simple proofs of recent results on the convexity of the integral of a set-valued function, and on the characterization of restricted cores of a saturated economy. (125 words)

Key words: Saturated measure space; Lyapunov's theorem; Maharam-type; integral of correspondences; measurable selection; Radon–Nikodym property, saturated economy, restricted cores.

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Contents

1 Introduction

Along with the Brouwer–Kakutani–Fan–Glicksberg fixed point theorems, and versions of the Hahn–Banach theorem, Lyapunov's theorem [42] on the range of an atomless finite-dimensional vector measure is a staple of modern mathematical economics, specifically general equilibrium and game theory; see the references furnished in [29]. To be sure, the relevance of the result goes beyond mathematical economics to several areas in applied mathematics, including statistical decision and optimal control theory. Indeed, the impetus for an infinite-dimensional generalization of the theorem has initially come from these quarters; see Kluvánek and Knowles [34, Chapters V and VI] and their references [33, 60, 58, 59, 35, 44, 45, 53]. These references spell out the trajectory of how, with an additional condition, the closure operator on the range in Uhl's approximate version of the Lyapunov theorem can be dropped, and the range space generalized from a Banach space with the Radon–Nikodym property (RNP) to a general Banach space. This additional condition used the concept of a *thin* set, based on a notion that can be traced to Kingman and Robertson [33] (also see [58]) in its formulation of a requirement on the non-injectivity of an operator; see Knowles' theorem in [34, Theorem V.1] and the discussion in Section 3 below.

It bears emphasis, however, that in mathematical economics, it is not so much Lyapunov's theorem itself that has proved to be the result of substantive consequence and use, but rather its straightforward corollary pertaining to the integral of a set-valued function (synonymously, correspondence) on an atomless probability space. It is this result that is *the* vehicle for the formalization of the intuition that "aggregation eliminates non-convexity", and thereby allows a substitution of an atomless multiplicity for the convexity assumption. However, this convexity property, while true for a correspondence on an atomless probability space with a finite-dimensional Euclidean space as its range, is only approximately true in general if (i) the range space is infinite-dimensional, or (ii) the domain consists of a "large but finite" index set; see [25, 27] and their references. This approximation testifies to the fact that Lyapunov's theorem is false if either of the twin assumptions of atomlessness and finite-dimensionality are dispensed with; see [60, 9]. These approximation results rely on the Shapley–Folkman theorem in the first instance, and on Uhl's approximate Lyapunov theorem in the second where it is the closure of the integral, rather than the integral itself, that is shown to be convex: see [27, 28, 46] for the latter case and [29] for references to the former.

It is thus not surprising that in the last decade and a half, there has been a sustained attempt in mathematical economics to eliminate the closure operator in the theory of integration of an atomless correspondence taking values in an infinite-dimensional Banach space. This has proceeded in what can now be seen as two well-identified steps: work on correspondences defined on (i) a Loeb probability space, first introduced in Loeb [38] (also see an exposition in [39]), (ii) a saturated probability space, first introduced Hoover-Keisler [19] (also see the comprehensive exposition in [14]) in the form of a saturated filtration to study systematically the existence of strong solutions for stochastic integral equations. There are two observations from [19] that are relevant in this connection: first, the saturation property can be directly connected to Maharam's [43] classification of measure algebras; and second, in the context of single measure space, Loeb spaces satisfy the property, and therefore can be regarded as a special case of a saturated space. Moreover, as stressed in Keisler-Sun [23, Acknowledgement], and also in [57] and [41, Remarks 2.5 and 2.6], results on Loeb spaces can be transferred in a straightforward way from Loeb spaces to saturated spaces. To be sure, once the connection to Maharam's work is made, and the saturated property identified, loosely speaking, as the requirement that the restriction of the σ-algebra to any set of positive measure be *not* countably-generated, one can ask whether Loeb spaces can be bypassed altogether, and the results such as those in [55, 56] proved directly from this identification. Such alternative direct proofs have been systematically presented in Podczeck [47, 48]. Thus, whatever the route, both approaches led to the same consequence and we now have a well-articulated theory of the convexity of the integral, rather than the closure of the integral, of a set-valued function taking values in an infinite-dimensional Banach space. What is of especial and key interest in all of this work is the demonstration that a saturated space is not only sufficient but also *necessary* for the result, as summed up by Keisler-Sun in their pithy aphorism that "any measure space that 'outperforms' the Lebesgue interval must be saturated."

A natural question pertains to the Lyapunov theorem itself: do its conclusions regarding the convexity and closure of the range hold for a vector measure taking values in an infinite-dimensional Banach space but defined on a saturated space, and dispensing with the non-injectivity condition of the earlier control-theory literature? We answer this question in this paper. We give a complete characterization of Lyapunov's theorem, one that identifies a saturated measure space to be necessary and sufficient for the validity of its conclusions in the case of an infinite-dimensional Banach space that is not necessarily separable nor has the RNP property. Our work takes off from the Knowles–Rudin condition on the non-injectivity of a suitable operator, as further utilized and emphasized in Rustichini-Yannelis [52] in terms of a requirement on the measure space. However this requirement has provoked the observation that "there is no natural measure space satisfying the required condition", see [56, Remark on pp.140–141]. Whereas this is incontrovertibly true, and the requirement does not have direct relevance to saturated spaces, it has an affinity to Condition 3.2 below on higher order Maharam-types that we utilize to derive the Lyapunov property in a setting that goes beyond separable Banach spaces. We may also note here that characterizations of the Lyapunov property in terms of sign embedding operators and "small enough" atoms are respectively available in [20, 21, 22, 53] and in [11]; and those of atomlessness in terms of strong continuity of a vector measure taking values in a Fréchet space in [61] and his references; but they are all phrased in terms of the closure operator. The point is that conditions on the measure space itself that are necessary and sufficient for the Lyapunov property to hold without this operator, as are furnished in this paper, have proved elusive.

We also present two applications of our principal result. The first focuses on the integration of correspondences, and offers a short direct proof of the convexity result recently established in [47, 57]. It is satisfying that the proof simply rehearses the standard argumentation for the finite-dimensional Euclidean case; see Debreu [6, pp. 369–370] and his references. The second application focuses on a Schmeidler's [54] remarkable application of the Lyapunov's theorem to show the irrelevance of a substantive restriction on the *core,* a solution concept for the allocation of resources of an exchange economy with an atomless continuum of agents; also the subsequent work [62, 16, 24, 13] and their references. Our result allows us to recover in its entirety Schmeidler's finite-dimensional, purely measure-theoretic intuition in an infinite-dimensional setting, and here again enables a rehearsal of the standard arguments without extraneous assumptions involving the closure operator through the assumption of continuous preferences.

The paper proceeds as follows: after collecting preliminaries in Section 2, and some preparatory results in Section 3, the main results are presented in Sections 4 and 5, with the two applications in Section 6.

2 Preliminaries

This section collects some basic notions and results employed in the sequel. We begin with the saturation property for finite measure spaces in terms of measure algebras, and observe that it reinforces the more conventional notion of atomlessness. Next, we define the notion of saturation for vector measures via their control measures with the saturated property.

2.1 Saturated Measure Spaces

A *measure algebra* is a pair $(\mathcal{F},\mu)$, where $\mathcal{F}$ is a Boolean σ-algebra with binary operations $\vee$ and $\wedge$, a unary operation ^{c} and a real-valued function $\mu:\mathcal{F}\to\mathbb{R}$ satisfying the following conditions: (i) $\mu(A)=0$ if and only if $A=\emptyset$, where $\emptyset=\Omega^c$ is the smallest element in $\mathcal{F}$ and $\Omega=\emptyset^c$ is the largest element in $\mathcal{F}$; (ii) $\mu(\bigvee_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}\mu(A_n)$ for every sequence $\{A_n\}$ in $\mathcal{F}$ such that $A_n\wedge A_m=\emptyset$ whenever $m\neq n$.

Let $(\mathcal{F},\mu)$ and $(\mathcal{G},\nu)$ be measure algebras. A mapping $\Phi:\mathcal{F}\to\mathcal{G}$ is an *isomorphism* if it is a bijection satisfying the following conditions: (i) $\Phi(A\wedge B^c)=\Phi(A)\wedge\Phi(B)^c$ for every $A,B\in\mathcal{F}$; (ii) $\Phi(A\vee B)=\Phi(A)\vee\Phi(B)$ for every $A,B\in\mathcal{F}$; (iii) $\mu(A)=\nu(\Phi(A))$ for every $A\in\mathcal{F}$. When such an isomorphism exists, $(\mathcal{F},\mu)$ is said to be *isomorphic* to $(\mathcal{G},\nu)$.

A *subalgebra* of $\mathcal{F}$ is a subset of $\mathcal{F}$ that contains Ω and is closed under the Boolean operations $\vee$, $\wedge$ and ^{c}. A subalgebra $\mathcal{U}$ of $\mathcal{F}$ is *order-closed* with respect to the order $\le$ given by $A \le B \iff A = A \wedge B$ if any nonempty upwards directed subsets of $\mathcal{U}$ with its supremum in $\mathcal{F}$ has the supremum in $\mathcal{U}$. A subset $\mathcal{U} \subset \mathcal{F}$ *completely generates* $\mathcal{F}$ if the smallest order closed subalgebra in $\mathcal{F}$ containing $\mathcal{U}$ is $\mathcal{F}$ itself. The *Maharam type* of $(\mathcal{F}, \mu)$ is the least cardinal number of any subset $\mathcal{U} \subset \mathcal{F}$ which completely generates $\mathcal{F}$.

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Denote by $L^1(\mu)$ the Banach space of μ-integrable functions on Ω and by $L^\infty(\mu)$ the Banach space of μ-essentially bounded functions on Ω. Let χ_E be the characteristic function of $E \in \mathcal{F}$. Denote by $L^1_E(\mu) = \{f\chi_E \mid f \in L^1(\mu)\}$ the vector subspace of $L^1(\mu)$ consisting of μ-integrable functions restricted to E and similarly, by $L^\infty_E(\mu) = \{f\chi_E \mid f \in L^\infty(\mu)\}$ the vector subspace of $L^\infty(\mu)$ consisting of μ-essentially bounded functions on Ω restricted to E.

For a finite measure space $(\Omega, \mathcal{F}, \mu)$, an equivalence relation $\sim$ on $\mathcal{F}$ is given by $A \sim B$ if and only if $\mu(A \triangle B) = 0$, where $A \triangle B$ is the symmetric difference of A and B in $\mathcal{F}$. The collection of equivalence classes is denoted by $\widehat{\mathcal{F}} = \mathcal{F}/\sim$ and its generic element $\widehat{A}$ is the equivalence class of $A \in \mathcal{F}$. The lattice operations $\vee$ and $\wedge$ in $\widehat{\mathcal{F}}$ are given in a usual way by $\widehat{A} \vee \widehat{B} = \widehat{A \cup B}$ and $\widehat{A} \wedge \widehat{B} = \widehat{A \cap B}$. The unary operation ^{c} in $\widehat{\mathcal{F}}$ is obtained for taking complements in $\widehat{\mathcal{F}}$ by $\widehat{A}^c = \widehat{(A^c)}$. Then $\widehat{\mathcal{F}}$ is an ordered set with its order $\le$. Under these operations $\widehat{\mathcal{F}}$ is a Boolean σ-algebra. Define the real-valued function $\hat{\mu} : \widehat{\mathcal{F}} \to [0, \infty)$ by $\hat{\mu}(\widehat{A}) = \mu(A)$ for $\widehat{A} \in \widehat{\mathcal{F}}$. Then the pair $(\widehat{\mathcal{F}}, \hat{\mu})$ is a measure algebra associated to $(\Omega, \mathcal{F}, \mu)$. The Maharam type of $(\Omega, \mathcal{F}, \mu)$ is defined to be that of $(\widehat{\mathcal{F}}, \hat{\mu})$.

We define a metric ρ on $\widehat{\mathcal{F}}$ by $\rho(\widehat{A}, \widehat{B}) = \mu(A \triangle B)$. Then $(\widehat{\mathcal{F}}, \rho)$ is a complete metric space (see [1, Lemma 13.13] or [10, Lemma III.7.1]). A measure algebra $(\widehat{\mathcal{F}}, \hat{\mu})$ is *separable* if $(\widehat{\mathcal{F}}, \rho)$ is a separable metric space. It is well known that $(\widehat{\mathcal{F}}, \hat{\mu})$ is separable if and only if $L^1(\mu)$ is separable (see [1, Lemma 13.14]).

Let $\mathcal{F}_E = \{A \cap E \mid A \in \mathcal{F}\}$ be a σ-algebra of $E \in \mathcal{F}$ inherited from $\mathcal{F}$ and define the subspace measure $\mu_E : \mathcal{F}_E \to [0, \infty)$ by $\mu_E(A) = \mu(A)$ for $A \in \mathcal{F}_E$. A finite measure space $(\Omega, \mathcal{F}, \mu)$ is (*Maharam-type-*)*homogeneous* if for every $E \in \mathcal{F}$ with $\mu(E) > 0$ the Maharam type of $(E, \mathcal{F}_E, \mu_E)$ is equal to that of $(\Omega, \mathcal{F}, \mu)$.

By the Maharam theorem (see [43, Theorem 1] or [15, Theorem 331I]), if $(\widehat{\mathcal{F}}, \hat{\mu})$ and $(\widehat{\mathcal{G}}, \hat{\nu})$ are homogeneous measure algebras of finite measure spaces with the same Maharam-type and the same mass, then $(\widehat{\mathcal{F}}, \hat{\mu})$ and $(\widehat{\mathcal{G}}, \hat{\nu})$ are isomorphic. The classical isomorphism theorem cited below (see [4, 17] and

[50, Theorem 15.3.4]) is a special case of the Maharam theorem.

Isomorphism Theorem. *Every separable measure algebra of an atomless probability space is isomorphic to the measure algebra of the Lebesgue unit interval.*

The notion of the saturation of measure spaces introduced below is a formalization of the property embodied by the product spaces of the form $\{0,1\}^{\mathfrak{m}}$ and $[0,1]^{\mathfrak{m}}$, where $\mathfrak{m}$ is an uncountable cardinal, $\{0,1\}$ has the uniform measure and $[0,1]$ has the Lebesgue measure (see [14, Theorem 3B.12] and [15, Theorem 331K]).

Definition 2.1. A finite measure space $(\Omega, \mathcal{F}, \mu)$ is *saturated* if for every $E \in \mathcal{F}$ with $\mu(E) > 0$ the Maharam type of $(E, \mathcal{F}_E, \mu_E)$ is uncountable. A measure μ is *saturated* if $(\Omega, \mathcal{F}, \mu)$ is saturated.

Atomless Loeb probability spaces are another example of saturated probability spaces (see [19]). Note that the measure algebras of $\{0,1\}^{\mathfrak{m}}$ and $[0,1]^{\mathfrak{m}}$ are isomorphic whenever $\mathfrak{m}$ is an infinite cardinal, and they are separable if and only if $\mathfrak{m}$ is countable. Thus, the countable products of $\{0,1\}$ and $[0,1]$ are typical examples of non-saturated atomless probability spaces. It is known in the literature that saturation is a much stronger condition on measure spaces than atomlessness. It is called "$\aleph_1$-atomless" in [19], "nowhere separable" in [12], "rich" in [23], "super-atomless" in [47] and "nowhere countably-generated" in [41]. For equivalent conditions on saturation, see [23] and their references.

Here, we employ the following useful characterization on atomlessness and saturation (see [47, Subsection 2.2(c) and Fact] and [15, 365X(p)]).

Proposition 2.1. (i) *A finite measure space $(\Omega, \mathcal{F}, \mu)$ is atomless if and only if for every $E \in \mathcal{F}$ with $\mu(E) > 0$ the Maharam type of $(E, \mathcal{F}_E, \mu_E)$ is infinite.*

(ii) *An atomless finite measure space $(\Omega, \mathcal{F}, \mu)$ is saturated if and only if $L^1_E(\mu)$ is nonseparable for every $E \in \mathcal{F}$ with $\mu(E) > 0$.*

2.2 Saturation and Vector Measures

Let $(\Omega, \mathcal{F})$ be a measurable space and X be a Banach space. A set function $m : \mathcal{F} \to X$ is *countably additive* if for every pairwise disjoint sequence $\{A_n\}$ in $\mathcal{F}$, we have $m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$, where the series is unconditionally convergent under the norm of X. Thanks to the Orlicz–Pettis theorem (see [9, Corollary I.4.4] or [10, Theorem IV.10.1]), a set function $m : \mathcal{F} \to X$ is countably additive if and only if x^*m is a finite signed measure on $\mathcal{F}$ for every $x^* \in X^*$. Throughout this paper, a countably additive set function $m : \mathcal{F} \to X$ is called a *vector measure*.

A set $A \in \mathcal{F}$ is an *atom* of m if $m(A) \neq \mathbf{0}$ and for every $E \in \mathcal{F}$ with $E \subset A$, either $m(E) = \mathbf{0}$ or $m(A \setminus E) = \mathbf{0}$. If m has no atom, it is said to be *atomless*. A set $N \in \mathcal{F}$ is *m-null* if $m(N \cap E) = \mathbf{0}$ for every $E \in \mathcal{F}$.

A vector measure $m : \mathcal{F} \to X$ is *absolutely continuous* with respect to a scalar measure μ if $\mu(A) = 0$ implies that $m(A \cap E) = \mathbf{0}$ for every $E \in \mathcal{F}$. A finite measure μ is a *control measure* of a vector measure $m : \mathcal{F} \to X$ whenever $\mu(A) = 0$ if and only if $m(A \cap E) = \mathbf{0}$ for every $E \in \mathcal{F}$. The Bartle–Dunford–Schwartz theorem guarantees that every vector measure in a Banach space possesses a control measure (see [3, Corollary 2.4], [9, Corollary I.2.6] or [10, Lemma IV.10.5]).

The significance of a control measure is exemplified by the observation that a set in $\mathcal{F}$ is an atom of m if and only if it is an atom of a control measure for m. This leads to the following characterization of the atomlessness of m.

Proposition 2.2. *Let μ be a control measure for a vector measure $m : \mathcal{F} \to X$. Then m is atomless if and only if for every $E \in \mathcal{F}$ with $\mu(E) > 0$ the Maharam type of $(E, \mathcal{F}_E, \mu_E)$ is infinite.*

Definition 2.2. A vector measure $m : \mathcal{F} \to X$ is *saturated* if its control measure μ is saturated.

In view of Proposition 2.2, the above definition obviously reinforces the atomlessness of vector measures. Saturated vector measures are atomless. Since every control measure for m is equivalent, it generates the same measure algebra on $\mathcal{F}$. Thus, the above definition is independent of the particular choice of the control measures for m. For the role of control measures in the analysis of the range of a vector measure, see [9, 22, 44].

3 An Operator-Theoretic Approach

In this section we present a systematic treatment of the range of vector measures in terms of the linear operators from $L^\infty(\mu)$ to X. The dependence of the validity of Lyapunov's theorem for infinite-dimensional Banach spaces on the non-injectivity of the suitable linear operator, and hence, on the dimensionality condition for $L^\infty(\mu)$ and X, is an important observation that can be traced to Rudin's [51] original reworking of Lindenstrauss' [37] finite-dimensional proof.

3.1 Lyapunov Measures and Lyapunov Operators

In this subsection we collect several results on the relation between Lyapunov measures and Lyapunov operators. Throughout this subsection, μ is a finite measure with respect to which a given vector measure $m : \mathcal{F} \to X$ is absolutely continuous. Note that except for Theorem 3.2 and Proposition 3.1, we do not necessarily assume that μ is a control measure of m.

A linear operator $T_m : L^\infty(\mu) \to X$ defined by $T_m f = \int f dm$ for $f \in L^\infty(\mu)$ is an *integral operator* of a vector measure m. The following continuity property of integral operators is well known (see [9, Lemma IX.1.3]).

Theorem 3.1. *The integral operator $T_m : L^\infty(\mu) \to X$ is continuous in the weak* topology for $L^\infty(\mu)$ and the weak topology for X.*

We first characterize atomless vector measures in terms of integral operators. To this end, we introduce the notion of atomless operators.

Definition 3.1. The integral operator $T_m : L^\infty(\mu) \to X$ is an *atomless operator* if for every $E \in \mathcal{F}$ with $\mu(E) > 0$ and $\varepsilon > 0$ there exists $f \in L^\infty_E(\mu) \setminus \{0\}$ with signed values $\{-1, 0, 1\}$ such that $\|T_m f\| < \varepsilon$.

Theorem 3.2. *Let μ be a control measure of a vector measure $m : \mathcal{F} \to X$. Then m is atomless if and only if $T_m : L^\infty(\mu) \to X$ is an atomless operator.*

Proof. Suppose that T_m is an atomless operator. If m has an atom $E \in \mathcal{F}$, then $m(E) \neq \mathbf{0}$ and E is an atom of μ. Then for every $\varepsilon > 0$ there exists $f \in L^\infty_E(\mu) \setminus \{0\}$ with signed values $\{-1, 0, 1\}$ such that $\|T_m f\| < \varepsilon$. Since measurable functions are constant on atoms, either $f = \chi_E$ or $f = -\chi_E$. We thus obtain $\|m(E)\| = \|T_m f\| < \varepsilon$ for every $\varepsilon > 0$, and hence, $m(E) = \mathbf{0}$, a contradiction.

Conversely, suppose that T_m is not an atomless operator. Then there exists $E \in \mathcal{F}$ with $\mu(E) > 0$ and $\varepsilon > 0$ such that $\|T_m f\| \geq \varepsilon$ for every $f \in L^1_E(\mu)$ with signed values $\{-1, 0, 1\}$. Thus, for every $A \in \mathcal{F}_E$, we have $\|T_m \chi_A\| = \|m(A)\| \geq \varepsilon$, and hence E is an atom of m. □

Definition 3.2. (i) A vector measure $m : \mathcal{F} \to X$ is a *Lyapunov measure* if for every $E \in \mathcal{F}$ the set $m(\mathcal{F}_E)$ is weakly compact and convex in X.

(ii) The integral operator $T_m : L^\infty(\mu) \to X$ is a *Lyapunov operator* of m if for every $E \in \mathcal{F}$ with $\mu(E) > 0$ the restriction $T_m : L^\infty_E(\mu) \to X$ is not injective.

If m has an atom $E \in \mathcal{F}$, then evidently, $m(\mathcal{F}_E)$ is not convex in X. Therefore, every Lyapunov measure is atomless. As the next result demonstrates, the atomlessness of vector measures is reinforced as well by the notion of Lyapunov operators.

Proposition 3.1. *Let μ be a control measure of a vector measure $m : \mathcal{F} \to X$. If $T_m : L^\infty(\mu) \to X$ is a Lyapunov operator, then it is an atomless operator.*

Proof. If m is not an atomless operator, then m has an atom $E \in \mathcal{F}$ by Theorem 3.2. Then $m(E) \neq \mathbf{0}$. Since every $f \in L^\infty_E(\mu)$ is constant on the atom E of μ, we have $T_m f = \int_E f dm = \alpha m(E)$ with $\alpha \in \mathbb{R}$, which implies that $T_m f = \mathbf{0}$ if and only if $f = 0$. Therefore, $T_m : L^\infty_E(\mu) \to X$ is an injection. $\square$

We next characterize Lyapunov measures in terms of Lyapunov operators. The proof of the following proposition was found in [9, Theorem IX.1.4].

Proposition 3.2. *A vector measure $m : \mathcal{F} \to X$ is a Lyapunov measure if and only if $T_m : L^\infty(\mu) \to X$ is a Lyapunov operator of m. The range of a Lyapunov measure m is given by:*

$$m(\mathcal{F}) = \{T_m f \in X \mid 0 \le f \le 1,\, f \in L^\infty(\mu)\}.$$

Next, we rework a celebrated example of Uhl's Uhl [60] which shows that the closure of the range of m is neither compact nor convex in $L^1(\mu)$.

Example 3.1. Let $(\Omega, \mathcal{F}, \mu)$ be the Lebesgue unit interval $[0,1]$ and define $m : \mathcal{F} \to L^1(\mu)$ by $m(A) = \chi_A$ for $A \in \mathcal{F}$. Then m is an atomless vector measure with a control measure of μ. By Proposition 3.2, the integral operator $T_m : L^\infty(\mu) \to L^1(\mu)$ is not a Lyapunov operator. This fact can be verified directly as follows.

Let f be a step function of the form $f = \sum_{i=1}^k \alpha_i \chi_{A_i}$, where $\{A_1, \dots, A_k\}$ is a mutually disjoint finite sequence in $\mathcal{F}$ and $\{\alpha_1, \dots, \alpha_k\}$ is a finite sequence of real numbers. It is easy to see that $T_m f = f$. For an arbitrarily given $f \in L^\infty(\mu)$, choose a sequence of step functions $\{f_k\}$ that converges to f in the norm topology of $L^\infty(\mu)$. Then $\{f_k\}$ converges to f as well in the weak* topology of $L^\infty(\mu)$. Since $T_m : L^\infty(\mu) \to X$ is continuous in the weak* topology for $L^\infty(\mu)$ and the weak topology for X by Theorem 3.1, we have $T_m f_k \to Tf$ weakly in $L^1(\mu)$. It follows from $L^\infty(\mu) \subset L^1(\mu)$ that $\{f_k\}$ converges to f both in the norm and weak topologies of $L^1(\mu)$.

Hence, $\langle T_m f - f, \varphi\rangle = \lim_k \langle T_m f_k - f_k, \varphi\rangle = 0$ for every $\varphi \in L^1(\mu)$ in view of $T_m f_k = f_k$ for each k. Therefore, we obtain $T_m f = f$ for every $f \in L^\infty(\mu)$. This means that T_m is not a Lyapunov operator, but an atomless operator.

Remark 3.1. The equivalence between Lyapunov measures and Lyapunov operators were established first by Kluvánek and Knowles [34] for the case where X is a quasicomplete locally convex space and then elaborated in [9] in the current simpler form when X is a Banach space. Kadets and Shekhtman [22] proposed an alternative notion of Lyapunov measures in which the closure of the range $m(\mathcal{F})$ is convex and characterized them in terms of the "sign-embedding" operators from $L^\infty(\mu)$ to X. For another characterization of Lyapunov measures in terms of the sign-embedding operators from $L^1(\mu)$ to X, see Kadets and Popov [21].

3.2 The Dimensionality Condition

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Denote by $ca(\mathcal{F}, \mu, X)$ be the space of X-valued countably additive set functions on $\mathcal{F}$ which are absolutely continuous with respect to μ. Denote by $\mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ the space of linear operators from $L^\infty(\mu)$ to X which are continuous in the weak* topology for $L^\infty(\mu)$ and the weak topology for X.

Definition 3.3. A linear operator $T : L^\infty(\mu) \to X$ is a *local injection* if there exist $E \in \mathcal{F}$ with $\mu(E) > 0$ such that the restriction $T : L^\infty_E(\mu) \to X$ is an injection.

Proposition 3.3. *There exists a linear bijection from $ca(\mathcal{F}, \mu, X)$ onto $\mathcal{L}(L^\infty_{w^*}(\mu), X_w)$.*

Proof. By Theorem 3.1, for every $m \in ca(\mathcal{F}, \mu, X)$ the integral operator $T_m : L^\infty(\mu) \to X$ is continuous in the weak* topology for $L^\infty(\mu)$ and the weak topology for X. Hence, the mapping $\Psi : ca(\mathcal{F}, \mu, X) \to \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ defined by $\Psi(m) = T_m$ for $m \in ca(\mathcal{F}, \mu, X)$ is a linear injection. Conversely, for given $T \in \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$, define $m \in ca(\mathcal{F}, \mu, X)$ by $m(A) = T\chi_A$ for $A \in \mathcal{F}$. To demonstrate the countable additivity of m, let $\{A_i\}$ be a mutually disjoint sequence in $\mathcal{F}$ and $A = \bigcup_{i=1}^\infty A_i$. Since $\sum_{i=1}^k \chi_{A_i}$ converges weakly* to χ_A in $L^\infty(\mu)$ as $k \to \infty$, we have:

$$\sum_{i=1}^k x^* m(A_i) = x^* T\left(\sum_{i=1}^k \chi_{A_i}\right) \to x^* T\chi_A = x^* m(A)$$

for every $x^* \in X^*$, where the weak convergence in X follows from the continuity of T with respect to the weak* and weak topologies. Therefore,

x^*m is a finite signed measure for every $x^* \in X^*$, and hence, m is countably additive by the Orlicz–Pettis theorem. It is obvious that m is absolutely continuous with respect to μ.

Define the linear mapping by $\Phi : \mathcal{L}(L^\infty_{w^*}(\mu), X_w) \to ca(\mathcal{F}, \mu, X)$ by $\Phi(T)(A) = T\chi_A$ for $T \in \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ and $A \in \mathcal{F}$. Let $\Phi(T) = m$ and consider the integral operator T_m. Then T_m coincides with T on the set of all characteristic functions. Since the linear span of the set of all characteristic functions is norm dense in $L^\infty(\mu)$, it is also weakly* dense in $L^\infty(\mu)$. Let $f \in L^\infty(\mu)$ be given arbitrary and $\{f_k\}$ be a sequence of step functions in $L^\infty(\mu)$ which converges weakly* to f. We then have $x^*T_m f_k = x^*Tf_k \to x^*Tf$ for every $x^* \in X^*$ as $k \to \infty$ because T is continuous for the weak* and weak topologies. Therefore, $T_m f = Tf$ for every $f \in L^\infty(\mu)$. This means that $\Psi \circ \Phi(T) = T$ for every $T \in \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$. Similarly, let $\Psi(m) = T$ and consider $m' \in ca(\mathcal{F}, \mu, X)$ given by $m'(A) = T\chi_A$ for $A \in \mathcal{F}$. Then $m'(A) = T_m\chi_A = m(A)$ for every $A \in \mathcal{F}$. This implies that $\Phi \circ \Psi(m) = m$ for every $m \in ca(\mathcal{F}, \mu, X)$. Therefore, $\Psi = \Phi^{-1}$ is a bijection. $\square$

Corollary 3.1. *Every $m \in ca(\mathcal{F}, \mu, X)$ is a Lyapunov measure if and only if every $T \in \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ is not a local injection.*

The *topological dimension* of a Banach space X is the least cardinal number corresponding to a subset of X with linear span norm dense in X. We denote by $\dim_{\mathrm{top}} X$ the topological dimension of X. Consider the topological dimensionality condition:

$$\dim_{\mathrm{top}} L^\infty_E(\mu) > \dim_{\mathrm{top}} X \quad \text{for every } E \in \mathcal{F} \text{ with } \mu(E) > 0. \tag{3.1}$$

As shown below, (3.1) is a sufficient condition for every $T \in \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ not to be a local injection.

Theorem 3.3. *If the topological dimensionality condition* (3.1) *is satisfied, then every $T \in \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ is not a local injection.*

Proof. Take any $E \in \mathcal{F}$ with $\mu(E) > 0$. Since $L^\infty_E(\mu)$ is densely spanned by the set of all characteristic functions, there exists a subset $\mathcal{G}$ of $\mathcal{F}_E$ with $\operatorname{card}\mathcal{G} = \dim_{\mathrm{top}} L^\infty_E(\mu)$ such that $L^\infty_E(\mu)$ is densely spanned by the set $\{\chi_A \mid A \in \mathcal{G}\}$. For every $T \in \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$, we have:

$$\begin{aligned} T(L^\infty_E(\mu)) = T(\overline{\mathrm{span}}\,\{\chi_A \mid A \in \mathcal{G}\}) &= T(\overline{\mathrm{span}}^{w^*}\{\chi_A \mid A \in \mathcal{G}\}) \\ &\subset \overline{T(\mathrm{span}\,\{\chi_A \mid A \in \mathcal{G}\})}^{w} \\ &= \overline{T(\mathrm{span}\,\{\chi_A \mid A \in \mathcal{G}\})} \end{aligned}$$

$$= \overline{\operatorname{span}\{T\chi_A \mid A \in \mathcal{G}\}},$$

where the second equality uses the fact that span $\{\chi_A \mid A \in \mathcal{G}\}$ is weakly* dense in $L^\infty_E(\mu)$, the third inclusion follows from the continuity of T, the forth equality employs the observation that weak closedness coincides with closedness on convex sets in X and the last equality follows from the linearity of T. Thus, the vector subspace $T(L^\infty_E(\mu))$ of X is densely spanned by the set $\{T\chi_A \mid A \in \mathcal{G}\}$, and hence, $\dim_{\mathrm{top}} T(L^\infty_E(\mu)) \le \dim_{\mathrm{top}} X$. If $T : L^\infty_E(\mu) \to X$ is an injection, then $\dim_{\mathrm{top}} T(L^\infty_E(\mu)) = \operatorname{card} \mathcal{G} = \dim_{\mathrm{top}} L^\infty_E(\mu) > \dim_{\mathrm{top}} X$, a contradiction. Therefore, T is not a local injection. $\square$

We turn to a canonical situation in the literature where the topological dimensionality condition (3.1) is satisfied: the case where $(\Omega, \mathcal{F}, \mu)$ is saturated and X is separable.

Theorem 3.4. *If $(\Omega, \mathcal{F}, \mu)$ is a saturated finite measure space and X is a separable Banach space, then the topological dimensionality condition* (3.1) *is satisfied.*

Proof. Let $(\Omega, \mathcal{F}, \mu)$ be saturated and X be separable. Since μ is atomless by Proposition 2.1(i), $L^1_E(\mu)$ is nonseparable for every $E \in \mathcal{F}$ with $\mu(E) > 0$ by Proposition 2.1(ii). Thus, the topological dimension of $L^1_E(\mu)$ is greater than the cardinality $\aleph_0$ of the set of natural numbers. Let $\mathfrak{m} = \dim_{\mathrm{top}} L^1_E(\mu)$. Then $L^\infty_E(\mu)$ is isomorphic to $L^\infty[0,1]^{\mathfrak{m}}$ (see [49, p. 221]) and we obtain:

$$\dim_{\mathrm{top}} L^\infty_E(\mu) = \dim_{\mathrm{top}} L^\infty[0,1]^{\mathfrak{m}} = \mathfrak{m}^{\aleph_0} \ge 2^{\aleph_0} = \mathfrak{c} > \aleph_0,$$

where the second equality exploits [49, Remark, p. 222] and $\mathfrak{c}$ is the cardinality of the continuum. On the other hand, the topological dimension of separable Banach spaces is at most $\aleph_0$. $\square$

Remark 3.2. By the *algebraic dimension* of a Banach space X, we mean the cardinality of a Hamel basis in X, which is denoted by $\dim_{\mathrm{alg}} X$ to discriminate it from $\dim_{\mathrm{top}} X$. Note that $\dim_{\mathrm{alg}} X \ge \dim_{\mathrm{top}} X$ for every Banach space X (see [36, p. 110]). Another sufficient condition for every linear operator $T \in \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ not to be a local injection is:

$$\dim_{\mathrm{alg}} L^\infty_E(\mu) > \dim_{\mathrm{alg}} X \quad \text{for every } E \in \mathcal{F} \text{ with } \mu(E) > 0. \tag{3.2}$$

This algebraic dimensionality condition was introduced first by Rudin [51, Theorem 5.5] in the proof of the Lyapunov theorem for the case where X is finite dimensional so as to replace the induction argument of the proof by

Lindenstrauss [37]. The significance of condition (3.2) was also recognized by Kluvánek and Knowles [34, Theorem V.2.1] when X is a locally convex space, and as emphasized in [52, (A1)]), it plays a crucial role for demonstrating the existence of equilibria for economies with a measure space of agents and an infinite-dimensional commodity space.

As observed by [56, p. 140] and [47, p. 840], imposing the algebraic dimensionality condition (3.2) rules out natural measure spaces. In particular, discrepancies between the topological and algebraic dimensions lead to the existence of a saturated measure space that violates (3.2) even if X is separable. To illustrate this point, let $[0,1]^{\mathfrak{c}}$ be the product space of the Lebesgue unit interval $[0,1]$. We then have $\dim_{\mathrm{top}} L^1[0,1]^{\mathfrak{c}} = \mathfrak{c}$ (see [49, Remark, p. 221]) and $\operatorname{card} L^\infty[0,1]^{\mathfrak{c}} = \dim_{\mathrm{top}} L^\infty[0,1]^{\mathfrak{c}} = \mathfrak{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$ (see [49, Remark, p. 222]). Thus, $\dim_{\mathrm{alg}} L^\infty[0,1]^{\mathfrak{c}} = \dim_{\mathrm{top}} L^\infty[0,1]^{\mathfrak{c}} = \mathfrak{c}$. On the other hand, if X is separable and infinite-dimensional, then $\aleph_0 = \dim_{\mathrm{top}} X \le \dim_{\mathrm{alg}} X = \mathfrak{c}$. We feel that this justifies the introduction of the topological dimensionality condition (3.1) in Theorem 3.4.

4 Lyapunov Measures in Separable Banach Spaces

In this section, we focus on separable Banach spaces, with the first subsection devoted to a sufficiency result, and the second to two necessity results. Our results testify to the relevance of the saturation property for the conclusion of Lyapunov's theorem.

4.1 Saturation and Lyapunov Measures: A Sufficiency Theorem

The next theorem states that the saturation of m is a sufficient condition for m to be a Lyapunov measure, or equivalently, for T_m to be a Lyapunov operator, whenever X is separable.

Theorem 4.1. *Let X be a separable Banach space. If $m : \mathcal{F} \to X$ is a saturated vector measure, then it is a Lyapunov measure with its range $m(\mathcal{F})$ given by:*

$$m(\mathcal{F}) = \{T_m f \in X \mid 0 \le f \le 1,\ f \in L^\infty(\mu)\}.$$

Proof. Since the topological dimensionality condition (3.1) is satisfied by Theorem 3.4, T_m is a Lyapunov operator of m by Theorems 3.1 and 3.3, and Proposition 3.2. The result follows again from Proposition 3.2. $\square$

Remark 4.1. A celebrated example of Lyapunov's exhibits the existence of an l^2-valued vector measure on the Lebesgue unit interval with a non-convex range (see [9, Example IX.1.1]). The Lebesgue unit interval is a typical example of an atomless probability space that is not saturated. This example then suggests that the saturation property is indispensable in guaranteeing the convexity of the range of a vector measure in infinite-dimensional Banach spaces.

4.2 Saturation and Lyapunov Measures: Necessity Theorems

Denote by $ca(\mathcal{F}, X)$ the vector space of X-valued countably additive set functions on $\mathcal{F}$ to X and by $ca_0(\mathcal{F}, \mu, X)$ the vector subspace of $ca(\mathcal{F}, X)$ whose elements have a control measure μ.

Lemma 4.1. *Let X be an infinite-dimensional Banach space. If an atomless finite measure space $(\Omega, \mathcal{F}, \mu)$ is not saturated, then there exists $E \in \mathcal{F}$ with $\mu(E) > 0$ and $m \in ca(\mathcal{F}_E, \mu_E, X)$ such that m is not a Lyapunov measure.*

Proof. Let $(\Omega, \mathcal{F}, \mu)$ be an atomless finite measure space that is not saturated. Then by Proposition 2.1(ii), there exists $E \in \mathcal{F}$ with $\mu(E) > 0$ such that $L^1_E(\mu)$ is separable. Without loss of generality, we may assume that μ is an atomless probability measure because it is finite. Since the measure algebra $(\widehat{\mathcal{F}_E}, \widehat{\mu_E})$ is separable, by the isomorphism theorem, it is isomorphic to the measure algebra $(\widehat{\mathcal{L}}, \hat{\lambda})$ of the Lebesgue measure space $(I, \mathcal{L}, \lambda)$ on the unit interval $I = [0, 1]$. Denote by $\Phi : \widehat{\mathcal{F}_E} \to \widehat{\mathcal{L}}$ the isomorphism from $(\widehat{\mathcal{F}_E}, \widehat{\mu_E})$ onto $(\widehat{\mathcal{L}}, \hat{\lambda})$. Then Φ is a σ-isomorphism, i.e., it satisfies $\Phi(\bigvee_{n=1}^{\infty} \widehat{A_n}) = \bigvee_{n=1}^{\infty} \Phi(\widehat{A_n})$ for every sequence $\{\widehat{A_n}\}$ in $\widehat{\mathcal{F}_E}$ (see [50, Problem 15.3.11]).

By virtue of [9, Corollary IX.1.6], there exists a vector measure $G : \mathcal{L} \to X$ of bounded variation and a set $J \subset I$ with $J \in \mathcal{L}$ such that the set $\{G(S \cap J) \mid S \in \mathcal{L}\}$ is not a weakly compact convex set in X and λ is a control measure for G. Define $\widehat{G} : \widehat{\mathcal{L}} \to X$ by $\widehat{G}(\widehat{S}) = G(S)$ for $\widehat{S} \in \widehat{\mathcal{L}}$. If $\widehat{S} = \widehat{S'}$, then $\lambda(S \triangle S') = 0$, and hence, $S \triangle S' \in \mathcal{L}$ is a G-null set. We then have $\widehat{G}(\widehat{S}) = G(S) = G(S') = \widehat{G}(\widehat{S'})$. Thus, $\widehat{G}$ is well defined. Define the vector measure $m : \mathcal{F}_E \to X$ by $m(A) = \widehat{G}(\Phi(\widehat{A}))$ for $A \in \mathcal{F}_E$. We claim that μ_E is a control measure for m, and hence, $m \in ca_0(\mathcal{F}_E, \mu_E, X)$. To this end, recall the construction of m and note the following equivalence for $A \in \mathcal{F}_E$:

$$\mu_E(A) = 0 \Longleftrightarrow \widehat{\mu_E}(\widehat{A}) = 0 \Longleftrightarrow \hat{\lambda}(\Phi(\widehat{A})) = 0$$

$$\begin{aligned}
&\Longleftrightarrow \lambda(S) = 0\ \forall S \in \Phi(\widehat{A}) \\
&\Longleftrightarrow G(S \cap J') = \mathbf{0}\ \forall S \in \Phi(\widehat{A})\ \forall J' \in \mathcal{L} \\
&\Longleftrightarrow \widehat{G}(\Phi(\widehat{A} \wedge \widehat{E'})) = \mathbf{0}\ \forall \widehat{E'} \in \widehat{\mathcal{F}_E} \\
&\Longleftrightarrow m(A \cap E') = \mathbf{0}\ \forall E' \in \mathcal{F}_E,
\end{aligned}$$

where the second equivalence employs the fact that Φ is measure-preserving and the forth equivalence follows from the fact that λ is a control measure for $G \in ca_0(\mathcal{L}, \lambda, X)$. □

We present here two necessity results on saturation. The first necessity theorem provides a "local" characterization of saturated measure spaces in terms of Lyapunov measures and Lyapunov operators.

Theorem 4.2. *The following conditions are equivalent:*

(i) $(\Omega, \mathcal{F}, \mu)$ *is saturated.*

(ii) *For any infinite-dimensional separable Banach space* X*, every* $m \in ca_0(\mathcal{F}_E, \mu_E, X)$ *is a Lyapunov measure for every* $E \in \mathcal{F}$ *with* $\mu(E) > 0$.

(iii) *For any infinite-dimensional separable Banach space* X*, the integral operator* $T_m : L_E^\infty(\mu) \to X$ *is a Lyapunov operator for every* $m \in ca_0(\mathcal{F}_E, \mu_E, X)$ *and* $E \in \mathcal{F}$ *with* $\mu(E) > 0$.

Proof. (i) ⇒ (ii): From the definition of saturation it follows that $(\Omega, \mathcal{F}, \mu)$ is saturated if and only if $(E, \mathcal{F}_E, \mu_E)$ is saturated for every $E \in \mathcal{F}$ with $\mu(E) > 0$. Let X be an arbitrary infinite-dimensional separable Banach space. Then every element in $ca_0(\mathcal{F}_E, \mu_E, X)$ is saturated for every $E \in \mathcal{F}$ with $\mu(E) > 0$. Hence, the result follows from Theorem 4.1.

(ii) ⇔ (iii): See Proposition 3.2.

(ii) ⇒ (i): Let X be an arbitrary infinite-dimensional separable Banach space and suppose that every element in $ca_0(\mathcal{F}_E, \mu_E, X)$ is a Lyapunov measure whenever $\mu(E) > 0$. Since every element in $ca_0(\mathcal{F}_E, \mu_E, X)$ is atomless, μ_E is also atomless for every $E \in \mathcal{F}$ with $\mu(E) > 0$. Therefore, μ is also atomless. If μ is not saturated, then by Lemma 4.1, there exists a non-Lyapunov measure in $ca_0(\mathcal{F}_E, \mu_E, X)$ for some $E \in \mathcal{F}$ with $\mu(E) > 0$, a contradiction. □

The second necessity theorem characterizes the saturation of all X-valued vector measures defined on a measurable space $(\Omega, \mathcal{F})$.

Theorem 4.3. *If* X *is an infinite-dimensional separable Banach space, then the following conditions are equivalent:*

(i) *Every* $m \in ca(\mathcal{F}, X)$ *is saturated.*

(ii) *Every* $m \in ca(\mathcal{F}, X)$ *is a Lyapunov measure.*

(iii) $T_m : L^\infty(\mu) \to X$ *is a Lyapunov operator for every* $m \in ca(\mathcal{F}, X)$, *where* μ *is a control measure for* m.

Proof. (i) $\Rightarrow$ (ii): See Theorem 4.1.

(ii) $\Rightarrow$ (i): Suppose that every element in $ca(\mathcal{F}, X)$ is a Lyapunov measure. If $m \in ca(\mathcal{F}, X)$ is not saturated, then its control measure μ is not saturated. Since m is atomless by Theorem 3.2, and Propositions 3.1 and 3.2, μ is also atomless. By Lemma 4.1, there exists a non-Lyapunov measure $m_E \in ca_0(\mathcal{F}_E, \mu_E, X)$ for some $E \in \mathcal{F}$ with $\mu(E) > 0$. Extend m_E from $\mathcal{F}_E$ to $\mathcal{F}$ by $\tilde{m}(A) = m_E(A \cap E)$ for $A \in \mathcal{F}$. Then $\tilde{m} \in ca(\mathcal{F}, X)$ is not a Lyapunov measure, a contradiction.

(ii) $\Leftrightarrow$ (iii): See Proposition 3.2. $\square$

Remark 4.2. One cannot remove the infinite-dimensionality of Banach spaces from Theorems 4.2 and 4.3. Suppose that X is a finite-dimensional Banach space. If $(\Omega, \mathcal{F}, \mu)$ is an atomless finite measure space with $\mathcal{F}$ countably generated, which is satisfied for the Lebesgue unit interval, then the Maharam type of $(\Omega, \mathcal{F}, \mu)$ is countable by Proposition 2.1(i), and consequently, μ is not saturated. If $m \in ca_0(\mathcal{F}, \mu, X)$, then the range $m(\mathcal{F})$ is compact and convex by the classical Lyapunov theorem. This leads to a counterexample of Theorems 4.2 and 4.3 whenever the infinite-dimensionality requirement is violated.

Remark 4.3. It is worth noting that in recent work, Khan and Zhang [32] mute somewhat the implication of the necessity result by demonstrating that Lyapunov's counterexample is no longer a counterexample for a countably-generated Lebesgue extension of the unit interval. Such an extension is of course not saturated, and yet furnishes the convexity of the range of the l^2-valued vector measure.

5 Lyapunov Measures in Non-Separable Banach Spaces

The aim this section is to provide a classification of Banach spaces for which Lyapunov's theorem holds in terms of the Maharam-types. It is now well-understood that the Maharam theorem identifies the basic "building blocks" for measure spaces, and our investigation relies on them to go beyond Lyapunov's theorem for separable Banach spaces established in Theorem 4.1 and opens the door to Lyapunov's theorem for their non-separable counterparts. We also present an exact version of the Lyapunov theorem for Banach spaces with the Radon–Nikodym property along the lines of [60].

5.1 Maharam-Types and the Lyapunov Property

Definition 5.1. A Banach space X has the *Lyapunov property* with respect to a finite measure space $(\Omega, \mathcal{F}, \mu)$ if every $m \in ca(\mathcal{F}, \mu, X)$ is a Lyapunov measure.

The next result reveals an ambivalence of the Lyapunov property in infinite-dimensional Banach spaces.

Proposition 5.1. *Every infinite-dimensional Banach space does not possess the Lyapunov property with respect to countably-generated, atomless probability spaces. However, for every Banach space there exists an atomless probability space with respect to which it has the Lyapunov property.*

Proof. It follows from the isomorphism theorem that the measure algebra of a countably-generated, atomless probability space is isomorphic to the measure algebra of the Lebesgue unit interval. The first part of the proposition draws from Uhl's celebrated example (already referred to above) whereby, for every infinite-dimensional Banach space X, there exists an X-valued vector measure defined on the Lebesgue unit interval such that its range is not convex in X.

For the second part, in view of Proposition 3.2, Corollary 3.1 and Theorem 3.4, it suffices to show that for every infinite-dimensional Banach space X, there exists a saturated probability space such that condition (3.1) is satisfied. Let $\mathfrak{m} = \dim_{\mathrm{top}} X$ and take any infinite cardinal $\mathfrak{n} > \mathfrak{m}$. Consider the probability space $(\Omega, \mathcal{F}, \mu)$ with the product space $\Omega = [0,1]^{\mathfrak{n}}$ of the Lebesgue unit interval. Since $[0,1]^{\mathfrak{n}}$ is isomorphic to $\{0,1\}^{\mathfrak{n}}$, by the Maharam theorem, $[0,1]^{\mathfrak{n}}$ is a Maharam-type-homogenous, of Maharam type $\mathfrak{n}$ (see [15, Theorem 331K]). That is, for every $E \in \mathcal{F}$ with $\mu(E) > 0$, the Maharam type of $(E, \mathcal{F}_E, \mu_E)$ is $\mathfrak{n}$. Without loss of generality, we may assume that μ_E is a probability measure on $\mathcal{F}_E$. Again by the Maharam theorem (see [15, Theorem 331I]), the measure algebra $(\widehat{\mathcal{F}}, \hat{\mu})$ of $(\Omega, \mathcal{F}, \mu)$ is isomorphic to the measure algebra $(\widehat{\mathcal{F}_E}, \widehat{\mu_E})$ of $(E, \mathcal{F}_E, \mu_E)$. This means that there is a linear isometry from $L^\infty(\mu)$ onto $L^\infty_E(\mu)$ for every $E \in \mathcal{F}$ with $\mu(E) > 0$ (see [15, Theorem 363F]). Since $\operatorname{card} L^\infty(\mu) = \dim_{\mathrm{top}} L^\infty(\mu) = \mathfrak{n}^{\aleph_0}$ (see [49, Remark, p. 222]), we obtain $\dim_{\mathrm{top}} L^\infty_E(\mu) = \mathfrak{n}^{\aleph_0} > \mathfrak{m} = \dim_{\mathrm{top}} X$. □

In our consideration of Maharam-types, we begin by presenting this basic benchmark.

Lemma 5.1. *Every countably-generated, atomless, finite measure space induces a homogeneous measure algebra of Maharam-type* $\aleph_0$*.*

Proof. By Proposition 2.1(i), the Maharam type of a countably-generated, atomless, finite measure space $(\Omega, \mathcal{F}, \mu)$ is countable. If $(\Omega, \mathcal{F}, \mu)$ were not Maharam-type-homogenous, then there should exist $E \in \mathcal{F}$ with $\mu(E) > 0$ such that the Maharam type of $(E, \mathcal{F}_E, \mu_E)$ is finite. This means that $(E, \mathcal{F}_E, \mu_E)$ has an atom, again by Proposition 2.1(i), a contradiction. $\square$

Denote by $(\Omega_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}}, \mu_{\mathfrak{m}})$ a finite measure space that induces a homogeneous measure algebra of Maharam-type $\mathfrak{m}$. Then $\mu_{\mathfrak{m}}$ is atomless whenever $\mathfrak{m} \geq \aleph_0$ and moreover by Lemma 5.1, the equality holds if $\mathcal{F}_{\mathfrak{m}}$ is countably generated. By definition, $(\Omega_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}}, \mu_{\mathfrak{m}})$ is saturated whenever $\mathfrak{m} \geq \mathfrak{c}$. By Theorem 4.1, if $\mathfrak{m} \geq \mathfrak{c}$, then separable Banach spaces have the Lyapunov property with respect to $(\Omega_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}}, \mu_{\mathfrak{m}})$. Thus, the minimum cardinality satisfying the Lyapunov property for infinite-dimensional separable Banach spaces is $\mathfrak{m} = \mathfrak{c}$ in view of Proposition 5.1.

When X is non-separable, condition (3.1) guarantees that the Lyapunov property for X is satisfied if $\mathfrak{m}^{\aleph_0} > \dim_{\mathrm{top}} X$ because $(\Omega_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}}, \mu_{\mathfrak{m}})$ is isomorphic to $[0,1]^{\mathfrak{m}}$ by the Maharam theorem and $\dim_{\mathrm{top}} L^\infty(\mu_{\mathfrak{m}}) = \mathfrak{m}^{\aleph_0}$ (see [49, Remark, p. 222]). Since the class of all cardinals is well-ordered with respect to cardinality, the minimum cardinal satisfying the Lyapunov property for X indeed exists.

For any Banach space X, let $\mathfrak{m}(X)$ be the minimal cardinality $\mathfrak{m}$ such that X has the Lyapunov property with respect to $(\Omega_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}}, \mu_{\mathfrak{m}})$. Then $\mathfrak{m}(X) = \aleph_0$ if X is a finite-dimensional Banach space; $\mathfrak{m}(X) = \mathfrak{c}$ if X is an infinite-dimensional separable Banach space; $\mathfrak{m}(X) \geq \mathfrak{c}$ if X is a non-separable Banach space.

We summarize the above observations through the following characterization of the Lyapunov property for Banach spaces.

Proposition 5.2. *A Banach space X has the Lyapunov property with respect to a Maharam-type-homogenous, finite measure space if and only if its Maharam type is greater than or equal to* $\mathfrak{m}(X)$*.*

In particular, for non-separable Banach spaces, we have the following estimate of $\mathfrak{m}(X)$.

Theorem 5.1. *If X is a non-separable Banach space with* $\dim_{\mathrm{top}} X = \mathfrak{m}$*, then* $\mathfrak{c} \leq \mathfrak{m}(X) \leq \mathfrak{c}^{\mathfrak{m}}$*.*

Proof. Let $(\Omega, \mathcal{F}, \mu)$ be a homogeneous finite measure space of Maharam type $2^{\mathfrak{m}}$. By the Maharam theorem, every measure space $(E, \mathcal{F}_E, \mu_E)$ with $\mu(E) > 0$ is isomorphic to the product space $[0,1]^{2^{\mathfrak{m}}}$ of the Lebesgue unit interval $[0,1]$. We thus have:

$$\dim_{\mathrm{top}} L^\infty_E(\mu) = (2^{\mathfrak{m}})^{\aleph_0} = (2^{\aleph_0})^{\mathfrak{m}} = \mathfrak{c}^{\mathfrak{m}} \geq 2^{\mathfrak{m}} > \mathfrak{m},$$

where the first equality follows from [49, Remark, p. 222]. Take any $m \in ca(\mathcal{F}, \mu, X)$. By Corollary 3.1 and Theorem 3.3, T_m is a Lyapunov operator. Therefore, $\mathfrak{m}(X) \leq \mathfrak{c}^{\mathfrak{m}}$. □

Corollary 5.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a Maharam-type-homogeneous, finite measure space. Then the following conditions are equivalent.*

(i) *X has the Lyapunov property with respect to $(\Omega, \mathcal{F}, \mu)$.*

(ii) *Every $T \in \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ is not a local injection.*

(iii) *The Maharam type of $(\Omega, \mathcal{F}, \mu)$ is greater than or equal to $\mathfrak{m}(X)$.*

5.2 Saturation and the Radon–Nikodym Property

Denote by $L^1(\mu, X)$ the Banach space of X-valued Bochner integrable functions g on Ω normed by $\|g\|_1 = \int \|g\| d\mu$. Uhl's [60] "approximate" version of the Lyapunov theorem for Banach spaces with the RNP has already been referred to, and for the reader's convenience we note that Banach space X has the *Radon–Nikodym property* (*RNP*) with respect to $(\Omega, \mathcal{F}, \mu)$ if for every vector measure $m : \mathcal{F} \to X$ of bounded variation that is absolutely continuous with respect to μ, there exists $g \in L^1(\mu, X)$ such that $m(A) = \int_A g d\mu$ for every $A \in \mathcal{F}$. A Banach space X has the *RNP* if it has the RNP with respect to every finite measure space.

We can now recall Uhl's theorem (see also [9, Theorem IV.1.10]): *Let X be a Banach space with the RNP. If $m \in ca(\mathcal{F}, X)$ is an atomless vector measure of bounded variation, then the norm closure of the range of m is norm compact and convex in X.* When the underlying vector measures are saturated, we obtain an "exact" version of the Lyapunov theorem for Banach spaces with the RNP, in which the closure operation is removed.

Theorem 5.2. *Let X be a Banach space with the RNP. If $m \in ca(\mathcal{F}, X)$ is a saturated vector measure of bounded variation, then the range of m is norm compact and convex in X.*

Proof. Since m is of bounded variation, its control measure μ is given by its variation. By the RNP of X, there exists a function $g \in L^1(\mu, X)$ such that $m(A) = \int_A g d\mu$ for every $A \in \mathcal{F}$. Since Bochner integrable functions are μ-essentially separably valued (see [9, Theorem II.1.2]), we may assume without loss of generality that g takes values in a separable Banach space X. By Theorem 4.1, the range $m(\mathcal{F})$ is weakly compact and convex in X. Since m is atomless, the norm closure of $m(\mathcal{F})$ is norm compact by Uhl's Theorem. Therefore, $m(\mathcal{F})$ is norm compact and convex in X because on convex sets in X norm closedness is equivalent to weak closedness. $\square$

Remark 5.1. For a discussion of the importance of the bounded variation assumption on vector measures in Uhl's theorem, see the discussion in [53] and in [21, Introduction and Theorem 1]. The bounded variation condition is needed in Theorem 5.2 to obtain a density function from the RNP, but the proof does not require the control measure to be the variation of the given vector measure. We also present below an alternative proof of Theorem 5.2 that revolves on these considerations.

6 Two Applications

In this final section, we turn to two applications that bring out the power of saturation property. The first applies our principal result to show the convexity of the integral of a correspondence under the saturation assumption and to present an alternative proof of Theorem 5.2; for details as to the importance of this result in mathematical economics and in control theory, see [6, 34, 5, 63, 18] and their references. The second applies our principal result to an application of Lyapunov's theorem in mathematical economics that has proved to be rather influential and leads us to richer notions of restricted cores allocations of saturated economies; see Schmeidler [54] and his followers [16, 62, 24, 13]. It bears emphasis that our primary motivation is to argue for the relevance Lyapunov's theorem on saturated spaces in applied work, and that comprehensive investigations of both applications in their substantive registers will be presented elsewhere.

6.1 On Convexity of the Set-Valued Integral

As before, $(\Omega, \mathcal{F}, \mu)$ is a finite measure space. Let $(X, \mathcal{B}_X)$ be the Borel σ-algebra of the Banach space $(X, \|\cdot\|)$. A correspondence $\Gamma : \Omega \to 2^X$ is *measurable* if its graph is $\mathcal{F} \otimes \mathcal{B}_X$-measurable. It is *integrably bounded* if there exists $\varphi \in L^1(\mu)$ such that $\|x\| \le |\varphi(\omega)|$ for every $x \in \Gamma(\omega)$ a.e.

$\omega \in \Omega$. If Γ is a measurable correspondence from a finite measure space $(\Omega, \mathcal{F}, \mu)$ to a separable Banach space X, then by the measurable selection theorem (see [5, Theorem III.22] or [1, Corollary 18.27]), there exists a measurable function $g : \Omega \to X$ such that $g(\omega) \in \Gamma(\omega)$ a.e. $\omega \in \Omega$. For the difficulties with measurable correspondences with values in non-separable Hilbert spaces, see [30].

Denote by $\mathcal{S}^1_\Gamma$ the set of all Bochner integrable selections of Γ, i.e.:

$$\mathcal{S}^1_\Gamma = \{g \in L^1(\mu, X) \mid g(\omega) \in \Gamma(\omega) \text{ a.e. } \omega \in \Omega\}.$$

The *integral* of Γ with respect to μ is defined by:

$$\int \Gamma d\mu = \left\{ \int g d\mu \in X \mid g \in \mathcal{S}^1_\Gamma \right\}.$$

We denote by $\mathcal{G}(\Omega, X)$ the space of correspondences from Ω to X with nonempty values. If $(\Omega, \mathcal{F}, \mu)$ is a complete finite measure space and $\Gamma \in \mathcal{G}(\Omega, X)$ is measurable, integrably bounded and weakly compact valued, then $\mathcal{S}^1_\Gamma$ is nonempty and relatively weakly compact in $L^1(\mu, X)$ (see [8, 47]). If, moreover, Γ is convex valued and X is separable, then by Diestel's theorem (see [7, 26]), $\mathcal{S}^1_\Gamma$ is weakly compact and convex in $L^1(\mu, X)$, and $\int \Gamma d\mu$ is weakly compact and convex in X (see [27, 63, 18]).

In [47, Theorem 1], the following result is proved: *If X is an infinite-dimensional Banach space and the integral $\int \Gamma d\mu$ is convex in X for every $\Gamma \in \mathcal{G}(\Omega, X)$, then $(\Omega, \mathcal{F}, \mu)$ is saturated.* We provide the converse of this result with a simple proof. We invite to the reader to compare our convexity result with that presented in the literature: with the availability of Theorem 4.1, the sufficiency proof simply mimics that for the case of the correspondence taking values in a finite-dimensional Euclidean space as in [6]; also see [56, 47, 57].

Lemma 6.1. *Let $g \in L^1(\mu, X)$ and define the vector measure $m : \mathcal{F} \to X$ by:*

$$m(A) = \int_A g d\mu, \quad A \in \mathcal{F}.$$

If $(\Omega, \mathcal{F}, \mu)$ is a saturated finite measure space, then m is saturated.

Proof. The countable additivity of m follows from [9, Theorem II.2.4]. Let λ be a control measure for m. It follows from the absolute continuity of m with respect to μ that $\mu(A) = 0$ implies $m(A \cap E) = \mathbf{0}$ for every $E \in \mathcal{F}$. Therefore, $\mu(A \triangle B) = 0$ implies $\lambda(A \triangle B) = 0$. This means that the measure algebra $(\widehat{\mathcal{F}}(\mu), \hat{\mu})$ of the measure space $(\Omega, \mathcal{F}, \mu)$ is a subalgebra of the measure algebra $(\widehat{\mathcal{F}}(\lambda), \hat{\lambda})$ of the measure space $(\Omega, \mathcal{F}, \lambda)$ and for every $E \in \mathcal{F}$, we have $\widehat{\mathcal{F}_E}(\mu) \subset \widehat{\mathcal{F}_E}(\lambda)$. Therefore, the Maharam type of $(\widehat{\mathcal{F}_E}(\lambda), \widehat{\lambda_E})$ is not less than that of $(\widehat{\mathcal{F}_E}(\mu), \widehat{\mu_E})$. This means that the Maharam type of $(\widehat{\mathcal{F}_E}(\lambda), \widehat{\lambda_E})$ is uncountable whenever the Maharam type of $(\widehat{\mathcal{F}_E}(\mu), \widehat{\mu_E})$ is uncountable. □

Theorem 6.1. *If $(\Omega, \mathcal{F}, \mu)$ is a saturated finite measure space, then for every Banach space X and $\Gamma \in \mathcal{G}(\Omega, X)$, the integral $\int \Gamma d\mu$ is convex in X.*

Proof. Suppose that μ is saturated. Let X be an arbitrary Banach space and $\Gamma \in \mathcal{G}(\Omega, X)$ be given arbitrary. Choose any $g_1, g_2 \in \mathcal{S}^1_\Gamma$ and $t \in [0,1]$. Since Bochner integrable functions are μ-essentially separably valued (see [9, Theorem II.1.2]), we may assume without loss of generality that g_1 and g_2 take values in a separable Banach space X. It suffices to show that there exists $g \in \mathcal{S}^1_\Gamma$ such that $\int g d\mu = \int (tg_1 + (1-t)g_2) d\mu$. To this end, define the vector measure $m : \mathcal{F} \to X \times X$ by:

$$m(A) = \left(\int_A g_1 d\mu, \int_A g_2 d\mu \right), \quad A \in \mathcal{F}.$$

Since m is saturated by Lemma 6.1, the range of m is convex by Theorem 4.1. Then there exists $A \in \mathcal{F}$ such that $m(A) = tm(\Omega)$, which is equivalent to $\int_A g_i d\mu = t \int g_i d\mu$ for each $i = 1, 2$. Define $g = g_1 \chi_A + g_2 \chi_{\Omega \setminus A}$. We then have $g \in \mathcal{S}^1_\Gamma$ and:

$$\int g d\mu = \int_A g_1 d\mu + \int_{\Omega \setminus A} g_2 d\mu = t \int g_1 d\mu + (1-t) \int g_2 d\mu.$$

□

To present an alternative proof of Theorem 5.2, we cite the following result from [57, Proposition 1] (see also [56, Theorem 2]).

Proposition 6.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a saturated finite measure space. If $\Gamma \in \mathcal{G}(\Omega, X)$ is integrably bounded and norm compact valued, then $\int \Gamma d\mu$ is norm compact in X.*

An Alternative Proof of Theorem 5.2. By the RNP of X, there is a function $g \in L^1(\mu, X)$ such that $m(A) = \int_A g d\mu$ for every $A \in \mathcal{F}$, where μ is a control measure of m. Define the correspondence $\Gamma : \Omega \to 2^X$ by $\Gamma(\omega) = \{g(\omega), \mathbf{0}\}$ for $\omega \in \Omega$. Then, Γ is obviously integrably bounded and norm compact valued. Moreover, any measurable selection of Γ is of the form $g\chi_A$ with $A \in \mathcal{F}$, and hence, $\int \Gamma d\mu = m(\mathcal{F})$. An appeal to Theorem 6.1 and Proposition 6.1 yields the norm compactness and convexity of $\int \Gamma d\mu$. □

This alternative proof leads us to record the validity of the following corollary.

Corollary 6.1. *Let $(\Omega, \mathcal{F}, \mu)$ a saturated finite measure space and X be a Banach space. If a vector measure $m : \mathcal{F} \to X$ has a representation $m(A) = \int_A g d\mu$ for every $A \in \mathcal{F}$ with $g \in L^1(\mu, X)$, then the the range of m is norm compact and convex.*

As brought out in Lindenstrauss' [37] proof for the finite-dimensional case, each component measure is absolutely continuous with respect to the sum of its variation, and the resulting density function allows a demonstration the convexity of the range to be a consequence of the convexity of the integral of a correspondence; see [56, Remark to Theorem 2]. In this context, Uhl's [60, p. 162] evaluation is worth keeping in mind.

> If X is allowed to be a general Banach space and m is an X-valued measure of bounded variation, then one can assert that the range of m is precompact and that, in the nonatomic case the closure of the range of m is convex if [...] m has the representation $m(A) = \int_A g d\mu$, $A \in \mathcal{F}$ for some measure μ and some measurable g with $\int \|g\| d\mu < \infty$. However, this restriction appears, to the author, to be too severe for a general result.

6.2 On Restricted Cores of a Saturated Economy

We present another application of Theorem 6.1 to a characterization of the restricted core allocations of a continuum economy proposed by Schmeidler [54] and Vind [62]. For the description of economies with a measure space of agents and an infinite-dimensional commodity space, we follow the formulation of [31].

To clarify the power of saturation, we present the following preliminary results for comparison.

Lemma 6.2. *Let $(\Omega, \mathcal{F}, \mu)$ be an atomless finite measure space, X be a Banach space, Y be a Banach space with the RNP with respect to $(\Omega, \mathcal{F}, \mu)$ and $m : \mathcal{F} \to Y$ be a vector measure of bounded variation that is absolutely continuous with respect to μ. If $\int_E f d\mu = \mathbf{0}$ for $f \in L^1(\mu, X)$ and $E \in \mathcal{F}$ with $\mu(E) > 0$, then for every $t \in [0, 1]$ there exists a sequence $\{F_n\}$ in $\mathcal{F}$ with $F_n \subset E$ for each n such that:*

$$\lim_{n\to\infty} \left(\int_{F_n} f d\mu, m(F_n), \mu(F_n) \right) = (\mathbf{0}, tm(E), t\mu(E)).$$

Proof. Since Y has the RNP regarding to $(\Omega, \mathcal{F}, \mu)$, there exists $g \in L^1(\mu, Y)$ such that $m(A) = \int_A g d\mu$ for every $A \in \mathcal{F}$. Define $\varphi \in L^1(\mu, X \times Y \times \mathbb{R})$ by $\varphi(\omega) = (f(\omega), g(\omega), 1)$ for $\omega \in \Omega$ and the atomless vector measure $\tilde{m} : \mathcal{F}_E \to X \times Y \times \mathbb{R}$ by $\tilde{m}(A) = \int_A \varphi d\mu$ for $A \in \mathcal{F}_E$. An appeal to Uhl's theorem [60, p. 162] guarantees that the closure of $\tilde{m}(\mathcal{F}_E)$ is convex in $X \times Y \times \mathbb{R}$. Take any $t \in [0, 1]$. Since $(\mathbf{0}, tm(E), t\mu(E)) = t\tilde{m}(E) + (1 - t)\tilde{m}(\emptyset)$ is in the closure of $\tilde{m}(\mathcal{F}_E)$, there exists a sequence $\{F_n\}$ in $\mathcal{F}_E$ such that $\tilde{m}(F_n) \to (\mathbf{0}, tm(E), t\mu(E))$. $\square$

Lemma 6.3. *Let $(\Omega, \mathcal{F}, \mu)$ be a saturated finite measure space, X be a Banach space, Y be a Banach space with the RNP with respect to $(\Omega, \mathcal{F}, \mu)$ and $m : \mathcal{F} \to Y$ be a vector measure of bounded variation that is absolutely continuous with respect to μ. If $\int_E f d\mu = \mathbf{0}$ for $f \in L^1(\mu, X)$ and $E \in \mathcal{F}$ with $\mu(E) > 0$, then for every $t \in [0, 1]$ there exists $F \in \mathcal{F}$ with $F \subset E$ such that:*

$$\left(\int_F f d\mu, m(F), \mu(F) \right) = (\mathbf{0}, tm(E), t\mu(E)).$$

Proof. Let φ be the function and $\tilde{m}$ be the vector measure respectively given in the proof of Lemma 6.2. Define $\Gamma \in \mathcal{G}(E, X \times Y \times \mathbb{R})$ by $\Gamma(\omega) = \{\varphi(\omega), \tilde{\mathbf{0}}\}$ for $\omega \in E$, where $\tilde{\mathbf{0}}$ is the origin of $X \times Y \times \mathbb{R}$. Since any measurable selection of Γ is of the form $\varphi\chi_A$ with $A \in \mathcal{F}_E$ and $\tilde{m}(A) = \int_A \varphi d\mu_E$ for every $A \in \mathcal{F}_E$, we have $\int_E \Gamma d\mu_E = \tilde{m}(\mathcal{F}_E)$. In view of the saturation of $(E, \mu_E, \mathcal{F}_E)$, an appeal to Theorem 6.1 guarantees that $\tilde{m}(\mathcal{F}_E)$ is convex in $X \times Y \times \mathbb{R}$. Take any $t \in [0, 1]$. Since $(\mathbf{0}, tm(E), t\mu(E)) = t\tilde{m}(E) + (1 - t)\tilde{m}(\emptyset)$, there exists $F \in \mathcal{F}_E$ such that $(\mathbf{0}, tm(E), t\mu(E)) = \tilde{m}(F)$ by the convexity of $\tilde{m}(\mathcal{F}_E)$. $\square$

Remark 6.1. The above lemmas are significant extensions of [54] to the case where the underlying space is an arbitrary Banach space. When $X = \mathbb{R}^n$ and $Y = \mathbb{R}$ in Lemma 6.2, $\{F_n\}$ is shown to be a constant sequence equal to some $F \in \mathcal{F}$ with $F \subset E$ applying the classical Lyapunov's theorem, which is the case precisely covered by [54]. Lemma 6.2 is also a generalization of [13, p.1188] who treat the case that X is a Banach space and $Y = \mathbb{R}$. Under the saturation assumption, the approximation result in Lemma 6.2 can be strengthened to the exact result in Lemma 6.3.

We can now turn to the substantive formulation of an economy. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space of *agents* with its generic element denoted by $a \in \Omega$. A *commodity space* X is a Banach space. A *consumption set* $\mathcal{X}(a)$ of each agent is described by a correspondence $\mathcal{X} : \Omega \to 2^X$ with $\mathcal{X}(a) \subset X$ for every $a \in \Omega$. A Bochner integrable function $f \in L^1(\mu, X)$ is an *allocation* if $f(a) \in \mathcal{X}(a)$ a.e. $a \in \Omega$. A *preference relation* on $\mathcal{X}(a)$ is a correspondence

$P_a : \mathcal{X}(a) \to 2^{\mathcal{X}(a)}$ such that $f(a) \notin P_a(f(a))$ for every allocation f. An *initial endowment* $e(a)$ is given by a function $e \in L^1(\mu, X)$ with $e(a) \in \mathcal{X}(a)$. An *economy* $\mathcal{E}$ is a quadruple $\mathcal{E} = [(\Omega, \mathcal{F}, \mu), \mathcal{X}, (P_a)_{a\in\Omega}, e]$. When $(\Omega, \mathcal{F}, \mu)$ is atomless, $\mathcal{E}$ is called an *atomless economy* and $(\Omega, \mathcal{F}, \mu)$ is saturated, it is called a *saturated economy*.

An allocation f for an economy $\mathcal{E}$ is *feasible* if $\int f d\mu = \int e d\mu$. A *coalition* is a set A in $\mathcal{F}$ with $\mu(A) > 0$. A coalition $A \in \mathcal{F}$ *blocks* a feasible allocation f if there exists a feasible allocation g such that $g(a) \in P_a(f(a))$ a.e. $a \in A$ and $\int_A g\mu = \int_A e d\mu$. Such a coalition A is called a *blocking coalition* to f. The set of all feasible allocations that no coalition in $\mathcal{F}$ can block is the *core* of the economy $\mathcal{E}$, denoted by $\mathcal{C}(\mathcal{E})$.

The following definition of the core with restricted coalitions is due to [24]. Let $\varepsilon \in (0,1)$ be given arbitrarily. A coalition $A \in \mathcal{F}$ is an *upper ε-coalition* (resp. a lower *ε-coalition*) if $\mu(A) \ge \varepsilon\mu(\Omega)$ (resp. $\mu(A) \le \varepsilon\mu(\Omega)$). The set of all feasible allocations that no upper ε-coalition (resp. lower ε-coalition) in $\mathcal{F}$ can block is the *upper ε-core* (resp. *lower ε-core*) of the economy $\mathcal{E}$, denoted by $\mathcal{C}^\varepsilon(\mathcal{E})$ (resp. $\mathcal{C}_\varepsilon(\mathcal{E})$). It follows from the definitions that $\mathcal{C}(\mathcal{E}) \subset \mathcal{C}^\varepsilon(\mathcal{E})$ and $\mathcal{C}(\mathcal{E}) \subset \mathcal{C}_\varepsilon(\mathcal{E})$ for every $\varepsilon \in (0,1)$ and $\varepsilon_1 < \varepsilon_2$ with $\varepsilon_1, \varepsilon_2 \in (0,1)$ implies $\mathcal{C}^{\varepsilon_1}(\mathcal{E}) \subset \mathcal{C}^{\varepsilon_2}(\mathcal{E})$ and $\mathcal{C}_{\varepsilon_2}(\mathcal{E}) \subset \mathcal{C}_{\varepsilon_1}(\mathcal{E})$.

Theorem 6.2. *Let $\mathcal{E} = [(\Omega, \mathcal{F}, \mu), \mathcal{X}, (P_a)_{a\in\Omega}, e]$ be a saturated economy. Then:*

(i) *If f is a feasible allocation that is blocked by a coalition $A \in \mathcal{F}$ via a feasible allocation g, then for every $\varepsilon \in (0,1)$ there exists a coalition $E \in \mathcal{F}$ with $E \subset A$ and $\mu(E) = \varepsilon\mu(A)$ such that E blocks f via g.*

(ii) $\mathcal{C}(\mathcal{E}) = \mathcal{C}_\varepsilon(\mathcal{E})$ *for every* $\varepsilon \in (0,1)$.

(iii) $\mathcal{C}(\mathcal{E}) = \bigcap_{\varepsilon\in(0,1)} \mathcal{C}^\varepsilon(\mathcal{E})$.

Proof. (i) Since $A \in \mathcal{F}$ is a blocking coalition to f, there exists a feasible allocation g such that $\int_A (g - e) d\mu = \mathbf{0}$. By Lemma 6.3, for every $\varepsilon \in (0,1)$ there exists a coalition $E \in \mathcal{F}$ with $E \subset A$ satisfying $\int_E (g - e) d\mu = \mathbf{0}$. Hence, f is blocked by the coalition E via the feasible allocation g.

(ii) Take any $\varepsilon \in (0,1)$. If $f \notin \mathcal{C}(\mathcal{E})$, then there exist a coalition $A \in \mathcal{F}$ and a feasible allocation g such that $g(a) \in P_a(f(a))$ a.e. $a \in A$ and $\int_A (g - e) d\mu = \mathbf{0}$. By the condition (i) above, there exists a subcoalition $E \subset A$ with $\mu(E) = \varepsilon\mu(A) \le \varepsilon\mu(\Omega)$ satisfying $\int_E (g - e) d\mu = \mathbf{0}$. Hence, f is blocked by the lower ε-coalition E via g, and thus $f \notin \mathcal{C}_\varepsilon(\mathcal{E})$. Therefore, $\mathcal{C}_\varepsilon(\mathcal{E}) \subset \mathcal{C}(\mathcal{E})$.

(iii) If $f \notin \mathcal{C}(\mathcal{E})$, then there exist a coalition $A \in \mathcal{F}$ and a feasible allocation g such that $g(a) \in P_a(f(a))$ a.e. $a \in A$ and $\int_A (g-e)d\mu = \mathbf{0}$. When $\mu(A) < \mu(\Omega)$, choose $\varepsilon \in (0,1)$ sufficiently small such that $\mu(A) \geq \varepsilon\mu(\Omega)$. Then f is blocked by the upper ε-coalition A via g, and hence $f \notin \mathcal{C}^{\varepsilon}(\mathcal{E})$. When $\mu(A) = \mu(\Omega)$, take any $\varepsilon \in (0,1)$. By the condition (i) above, there exists a subcoalition $E \subset A$ with $\mu(E) = \varepsilon\mu(A)$ satisfying $\int_E (g-e)d\mu = \mathbf{0}$. Hence, f is blocked by the upper ε-coalition E via g, and thus $f \notin \mathcal{C}^{\varepsilon}(\mathcal{E})$. Therefore, $\bigcap_{\varepsilon\in(0,1)} \mathcal{C}^{\varepsilon}(\mathcal{E}) \subset \mathcal{C}(\mathcal{E})$. □

Condition (i) implies that if A is a blocking coalition to a feasible allocation f via a feasible allocation g, then there is an arbitrary small blocking coalition $E \subset A$ with $\mu(E) < \varepsilon$ to the same feasible allocation f also via g. Condition (ii) means that for every positive number ε, the core coincides with the set of all feasible allocations that are not blocked by any coalition of measure less than ε. Condition (iii) is the continuity property of the upper ε-core in the sense that $\mathcal{C}^{\varepsilon}(\mathcal{E})$ is monotone decreasing as $\varepsilon \uparrow 1$ and converges to $\mathcal{C}(\mathcal{E})$.

Remark 6.2. We stress that Theorem 6.2 retains the original spirit of [54], and that, unlike the treatment in [13] which involves an approximation argument based on the special case of Lemma 6.2, there is no additional assumption on the underlying data of the economy: the commodity space, consumption sets, preferences and initial endowments. However, note the asymmetry between the upper and lower cases of the ε-core, and one needs further assumptions to strengthen the "shrinkage" result reported here to one asserting the coincidence of the upper and exact cases, as in [62] in the finite-dimensional setting.

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