

Expected Multi-Utility Representations*

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December 13, 2011

Abstract

This paper axiomatizes expected multi-utility representations of incomplete preferences under risk and under uncertainty. The von Neumann-Morgenstern expected utility model with incomplete preferences is revisited using a “constructive” approach, as opposed to earlier treatments that use convex analysis.

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Keywords: Incomplete preferences, Multi-utility representations

“Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from a normative viewpoint.” (Aumann [1962], p. 446).

*We are grateful to Juan Dubra, Robert Nau and Teddy Seidenfeld for their useful comments. Part of this work was done when Edi Karni was visiting the ENS de Cachan.

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1 Introduction

The presumption that a decision maker is always able to express clear preference among alternatives has long been recognized as highly unrealistic, especially in situations requiring choice among alternatives involving distinct attributes that are not readily comparable.¹ Consequently, the use of the completeness axiom to model rational decision making is problematic, to say the least.

Aumann (1962) was the first to broach this issue in the context of expected utility theory under risk. Taking, as primitive, an incomplete transitive and reflexive binary relation on the set of risky prospects that satisfies the independence axiom and a weak form of continuity, Aumann showed that one risky prospect is (weakly) preferred over another only if its expected utility is greater for a set of von Neumann-Morgenstern utility functions.

Shapley and Baucells (1998) provided an axiomatic characterization of expected multi-utility representation of incomplete, transitive, and reflexive preference relation on a closed and convex subset of finite dimensional linear space satisfying independence and mixture continuity.²

Dubra, Maccheroni and Ok (2004) axiomatized expected multi-utility representation of incomplete preference relations over probability distributions, whose support is a compact metric space. They also showed that the utility functions that figure in the representation are unique in the sense that any other expected multi-utility representation of the same preference relation must span a cone whose closure is the same as that of the original representation with possible shifts resulting from adding constant functions.

In this note we revisit the von Neumann-Morgenstern model to study necessary and sufficient conditions for the existence of expected multi-utility representation and to study its uniqueness properties.³ Like Shapley and Baucells (1998), we assume that the choice set is a closed and convex subset of finite dimensional linear space. In addition, we assume that the choice set have a greatest element (that is, an element that is strictly preferred to every other element of the set) and a smallest element (that is, an element that every other element of the set is strictly preferred to it). Unlike Aumann

¹See von Neumann and Morgenstern (1947).

²See Baucells and Shapley (2008) for a brief summary of their 1998 result.

³Expected multi-utility representations of incomplete preferences under risk and additively separable multi-utility representation of incomplete preferences under uncertainty are obtained as corollaries of our main result.

(1962) and Shapley and Baucells (1998), our primitive preference relation is a strict partial order (that is, a transitive and irreflexive binary relation). To model the weak preference relation we invoke the definition of the closure of the strict preference relation introduced in Galaabaatar and Karni (2011). The difference between the definitions of the weak preferences, to be discussed in greater details below, highlights the distinction between the continuity assumption invoked by Shapley and Baucells (1998) and in our model. In particular, Shapley and Baucells assume mixture continuity. Consequently, incompleteness in their model imply that the Archimedean axiom cannot hold.⁴ By contrast, the two notions of continuity, namely, mixture and Archimedean continuity, are consistent with incomplete preferences in our model.⁵ Finally, Shapley and Baucells (1998) did not address the issue of uniqueness of the representation. In this paper we include the uniqueness result.

Unlike previous studies that use convex analysis as the main analytical tool, we take a “constructive” approach, which makes the representation more transparent and easier to understand. We also show that the expected multi-utility representation of the strict partial order extends to the weak partial order.

2 The von Neumann-Morgenstern Theory without the Completeness Axiom

2.1 The analytical framework and the preference structure

Let C be a convex subset of a suitably finite dimensional linear space, $\mathcal{L}$. Without loss of generality let $\mathcal{L}$ be chosen so that C and $\mathcal{L}$ have the same cardinal dimension.⁶ A *preference relation* is a binary relation on C denoted by $\succ$. The set C is said to be $\succ$ -*bounded* if there exist p^M and p^m in C such that $p^M \succ p \succ p^m$, for all $p \in C - \{p^M, p^m\}$.

Consider the following axioms depicting the structure of $\succ$.

⁴See Dubra (2011).

⁵See Karni (2011).

⁶This procedure which Shapley and Baucells (1998) refer to as efficient embedding, entails no essential loss of generality.

- (A.1) (Strict partial order)** The preference relation $\succ$ is transitive and irreflexive.
- (A.2) (Archimedean)** For all $p, q, r \in C$, if $p \succ q$ and $q \succ r$ then $\beta p + (1 - \beta) r \succ q$ and $q \succ \alpha p + (1 - \alpha) r$ for some $\alpha, \beta \in (0, 1)$.
- (A.3) (Independence)** For all $p, q, r \in C$ and $\alpha \in (0, 1]$, $p \succ q$ if and only if $\alpha p + (1 - \alpha) r \succ \alpha q + (1 - \alpha) r$.

The difference between the preference structure above and that of expected utility theory is that the induced relation $\neg(p \succ q)$ is reflexive but not necessarily transitive (hence it is not necessarily a preorder). Moreover, it is not necessarily complete. Thus, $\neg(p \succ q)$ and $\neg(q \succ p)$ does not imply that p and q are indifferent (i.e., equivalent), rather they may be *noncomparable*. If p and q are noncomparable we write $p \bowtie q$.

Definitions 1: For all $p, q \in C$, (a) $p \succsim_{GK} q$ if $r \succ p$ implies $r \succ q$, for all $r \in C$, (b) $p \sim q$ if $p \succsim_{GK} q$ and $q \succsim_{GK} p$; and (c) $p \succeq q$ if $p \succsim_{GK} q$ and $\neg(p \succ q)$.

If $\succ$ satisfies (A.1)-(A.3) then the derived binary relation $\succsim_{GK}$ on C is a weak order (that is, transitive and reflexive) satisfying the Archimedean and independence axioms that is not necessarily complete.⁷ The indifference relation, $\sim_{GK}$, that is, the symmetric part of $\succsim_{GK}$, is an equivalence relation.

Taking the strict preference relation, $\succ$, as primitive, it is customary to define a weak preference relations as the negation of $\succ$. Formally, given a binary relation $\succ$ on C , define a binary relation $\succsim$ on C by: $p \succsim q$ if $\neg(q \succ p)$.⁸ If the strict preference relation, $\succ$, is transitive and irreflexive, then the weak preference relation, $\succsim$, is complete. Karni (2011) shows that that weak preference relations $\succsim_{GK}$ and $\succsim$ agree if and only if $\succ$ is negatively transitive and $\succsim_{GK}$ is complete.

The standard practice in decision theory is to take the weak preference relation as primitive and define the strict preference relation as its asymmetric part. Invoking the standard practice, Dubra (2010), showed that if C is the set of lotteries on a finite set of prizes and the weak preference relation is nontrivial (that is, $\succ \neq \emptyset$) and satisfies (A.3), then any two of the following

⁷The proof of the claim about the independence is part of the proof of Theorem 1, below. Note that $\succ$ is not the asymmetric part of $\succsim$.

⁸For example, and Kreps (1988).

axioms implied the third, completeness, Archimedean, and mixture continuity.⁹ Thus, a nontrivial, partial, preorder satisfying independence must fail to satisfy one of the continuity axioms. Karni (2011) showed that, if the weak preference relation is as in Definitions 1, then a nontrivial preference relation may satisfy independence, Archimedean, mixture continuity and yet be incomplete. Hence, the approach taken here seems more natural for modeling incomplete preferences as an extension of choice theory with complete preferences.

For every $p \in C$, let $B(p) := \{q \in C \mid q \succ p\}$ and $W(p) := \{q \in C \mid p \succ q\}$ denote the upper and lower contour sets of p , respectively. The relation $\succ$ is convex if the upper contour set is convex.

Lemma 1: Let $\succ$ be a binary relation on C . If $\succ$ satisfies (A.1), (A.2) and (A.3) then it is convex. Moreover, the lower contour set is also convex.

The proof is by two applications of (A.3).¹⁰

2.2 The fundamental representation theorem

We present a general result giving rise to the finite-dimensional expected multi-utility representations under risk, and additively separable multi-utility representation under uncertainty, as immediate implications. To state this result, we use the following notations: Let $\mathcal{B}$ be a set of sets of real-valued, affine, functions on $\mathcal{L}$ such that $\mathcal{U} \in \mathcal{B}$ implies that $B(\mathcal{U}) := \{p \in \mathcal{L} \mid u(p) > u(0) \text{ for all } u \in \mathcal{U}\}$ is algebraically open in $\mathcal{L}$.

The Main Representation Theorem: Let C be a nonempty, convex, subset of a finite dimensional linear space, $\mathcal{L}$. Let $\succ$ be a binary relation on C , then the following conditions are equivalent:

- (i) C is $\succ$ -bounded and $\succ$ satisfies (A.1), (A.2) and (A.3)
- (ii) There exists nonempty closed convex set, $\mathcal{U} \in \mathcal{B}$, such that $U(p^M) > U(p) > U(p^m)$, for all $p \in C - \{p^M, p^m\}$ and $U \in \mathcal{U}$, and, for all $p, q \in C$,

$$q \succsim_{GK} p \Leftrightarrow U(q) \geq U(p) \text{ for all } U \in \mathcal{U} \quad (1)$$

⁹A weak preference relation satisfies mixture continuity if, for all $p, q, r \in \Delta(X)$ the sets $\{\alpha \in [0, 1] \mid \alpha p + (1 - \alpha)q \succsim r\}$ and $\{\alpha \in [0, 1] \mid r \succsim \alpha p + (1 - \alpha)q\}$ are closed.

¹⁰Let $q, r \in B(p)$ and $\alpha \in [0, 1]$. To prove the lemma we need to show that $\alpha q + (1 - \alpha)r \succ p$. Apply (A.3) twice to obtain, $\alpha q + (1 - \alpha)r \succ \alpha p + (1 - \alpha)r$ and $\alpha p + (1 - \alpha)r \succ \alpha p + (1 - \alpha)p$. The same applies to $W(p)$.

and

$$q \succ p \Leftrightarrow U(q) > U(p) \text{ for all } U \in \mathcal{U}. \quad (2)$$

Remark 1: It is shown in the proof and $q \succeq p$ if and only if $U(q) \geq U(p)$ for all $U \in \mathcal{U}$ and $U(q) = U(p)$ for some $U \in \mathcal{U}$.

Remark 2: Seidenfeld et. al. (1995) show that a strict partial order, defined by strict first-order stochastic dominance, has expected multi-utility representation, satisfies the independence axiom and violates the Archimedean axiom.¹¹ To bypass this problem, Seidenfeld et. al. (1995) and subsequent writers invoked alternative continuity axioms that, unlike the Archimedean axiom, require the imposition of a topological structures.¹² We maintained the Archimedean axiom as our continuity postulate at the cost of restricting the upper contour sets associated with the strict preference relation, $B(p) := \{q \in C \mid q \succ p\}$, to be algebraically open. (In the example of Seidenfeld et. al. (1995) these sets are closed). Given the trade-off involved, this restriction seems, to us, reasonable. Moreover, restricting the upper contour set in this manner, we follow a long tradition in economic theory.

The next theorem specifies the uniqueness of the representation. To state the uniqueness result, Following Dubra et. al. (2004) we denote by $\langle \mathcal{U} \rangle$ the closure of the convex cone generated by all the functions in $\mathcal{U}$ and all the constant function on $\mathcal{L}$.

Uniqueness Theorem: If $\mathcal{V}$ is another set in $\mathcal{B}$ that represents $\succsim_{GK}$ and $\succ$ in the sense of (1) and (2), respectively, then $\langle \mathcal{V} \rangle = \langle \mathcal{U} \rangle$.

2.3 Expected multi-utility representation for simple probability measures

Let $X = \{x_1, \dots, x_n\}$ be a finite set of prizes and denote by $\Delta(X)$ the set of all probability measures on X . For each $\ell, \ell' \in \Delta(X)$ and $\alpha \in [0, 1]$ define $\alpha\ell + (1 - \alpha)\ell' \in \Delta(X)$ by $(\alpha\ell + (1 - \alpha)\ell')(x) = \alpha\ell(x) + (1 - \alpha)\ell'(x)$, for all $x \in X$. Then $\Delta(X)$ is a convex subset of the linear space $\mathbb{R}^X$. Let $\ell^M, \ell^m \in \Delta(X)$ satisfy $\ell^M \succ \ell \succ \ell^m$, for all $\ell \in \Delta(X)$. Application of Theorem 1 to $C = \Delta(X)$ yields an expected multi-utility representation.

¹¹See example 2.1 in their paper.

¹²See Dubra et. al. (2004) and Nau (2006).

Corollary 1 (Expected multi-utility representation) Let $\succ$ be a binary relation on $\Delta(X)$, then the following conditions are equivalent:

- (i) $\Delta(X)$ is $\succ$ -bounded and $\succ$ satisfies (A.1), (A.2) and (A.3).
- (ii) There exists nonempty, closed and convex set, $\mathbb{U} \in \mathcal{B}$, of real-valued functions on X such that

$$\sum_{x \in \text{Supp}(\ell^M)} u(x) \ell^M(x) > \sum_{x \in \text{Supp}(\ell)} u(x) \ell(x) > \sum_{x \in \text{Supp}(\ell^m)} u(x) \ell^m(x),$$

for all $\ell \in \Delta(X) - \{\ell^M, \ell^m\}$, and $u \in \mathbb{U}$ and, for all $p, q \in \Delta(X)$,

$$p \succ_{GK} q \Leftrightarrow \sum_{x \in \text{Supp}(p)} u(x) p(x) \geq \sum_{x \in \text{Supp}(q)} u(x) q(x), \text{ for all } u \in \mathbb{U}, \quad (3)$$

and

$$p \succ q \Leftrightarrow \sum_{x \in \text{Supp}(p)} u(x) p(x) > \sum_{x \in \text{Supp}(q)} u(x) q(x), \text{ for all } u \in \mathbb{U}. \quad (4)$$

Moreover, if $\mathbb{V}$ is another set of real-valued, affine, functions on $\Delta(X)$ that represent $\succ_{GK}$ and $\succ$ in the sense of (3) and (4), respectively, then $\langle \mathbb{V} \rangle = \langle \mathbb{U} \rangle$.

Proof: Let $C = \Delta(X)$ and $\mathbb{U} = \{u \in \mathbb{R}^X \mid u \cdot p = U(p), U \in \mathcal{U}\}$, then the conclusions of the corollary are implied by Theorem 1.

2.4 Additively separable multi-utility representation

Consider the Anscombe-Aumann (1963) model. Let S be a finite set of *states*. Subsets of S are *events*. Let $H := \{h : S \rightarrow \Delta(X)\}$ be the set whose elements are *acts*. For all $h, h' \in H$ and $\alpha \in [0, 1]$, $\alpha h + (1 - \alpha) h' \in H$ is defined by $(\alpha h + (1 - \alpha) h')(s) = \alpha h(s) + (1 - \alpha) h'(s)$, for all $s \in S$.¹³ Under this definition $H = \Delta(X)^S$ is a convex subset of the linear space $\mathbb{R}^{X \times S}$. Let $h^M, h^m \in H$ satisfy $h^M \succ h \succ h^m$, for all $h \in H$.

Applying Theorem 1 to H , we obtain the following:¹⁴

¹³For every $s \in S$, the convex mixture $\alpha h(s) + (1 - \alpha) h'(s)$ is defined as in the preceding subsection.

¹⁴A similar result appears in Nau (2006) for finite X . Ok, et. al. (2008) show that the same holds when X is a compact metric space. As mentioned, these authors use a continuity assumption stronger than (A.2).

Corollary 2 (Additive multi-utility representation) Let $\succ$ be a binary relation on H , then the following conditions are equivalent:

- (i) H is $\succ$ -bounded and $\succ$ satisfies (A.1), (A.2) and (A.3).
- (ii) There exists nonempty, convex and closed, set $\mathcal{W} \in \mathcal{B}$ of real-valued functions, w , on $\Delta(X) \times S$, affine in its first argument such that

$$\sum_{s \in S} w(h^M(s), s) > \sum_{s \in S} w(h(s), s) > \sum_{s \in S} w(h^m(s), s),$$

for all $h \in H - \{h^M, h^m\}$, and $w \in \mathcal{W}$ and, for all $h, h' \in H$,

$$h \succ_{GK} h' \Leftrightarrow \sum_{s \in S} w(h(s), s) \geq \sum_{s \in S} w(h'(s), s), \text{ for all } w \in \mathcal{W}, \quad (5)$$

and

$$h \succ h' \Leftrightarrow \sum_{s \in S} w(h(s), s) > \sum_{s \in S} w(h'(s), s), \text{ for all } w \in \mathcal{W}. \quad (6)$$

Moreover, if $\mathcal{W}'$ is another set of real-valued, affine, functions on H that represent $\succ_{GK}$ and $\succ$ in the sense of (5) and (6), respectively, then $\langle \mathcal{W}' \rangle = \langle \mathcal{W} \rangle$.

The proof is by the application of a standard argument (see Kreps [1988]) to U in the proof of Theorem 1.

Remark 3: Let $\mathcal{W}_s := \{w(\cdot, s) \mid w \in \mathcal{W}\}$. By Corollary 1, $w(h(s), s) = \sum_{x \in \text{Supp}(h(s))} u(x; s) h(x; s)$, where $u(\cdot, s)$ is a real-valued function on X , for all $s \in S$.

The representations in Corollary 2 are not the most parsimonious as the set $\mathcal{W}$ includes functions that are redundant (that is, their removal does not affect the representation).

3 Proofs of the Main Theorem

3.1 Proof of the main representation theorem

(i) $\Rightarrow$ (ii). If $\succ$ is empty then, by definition, $p \succ_{GK} q$ and $q \succ_{GK} p$, for all $p, q \in C$. Let $\mathcal{U}$ be the set of all constant real-valued functions on C .

Henceforth, assume that $\succ$ is not empty.

A preference relation $\succ$ is said to satisfy *mixture monotonicity* if $p \succ q$ and $0 \leq \alpha < \beta \leq 1$ imply that $\beta p + (1 - \beta) q \succ \alpha p + (1 - \alpha) q$.

Claim 1: $\succ$ satisfies mixture monotonicity.

The proof, by standard argument, is an implication of (A.3).¹⁵

Claim 2. $\succ_{GK}$ satisfies independence (that is, for all $p, q, r \in C$ and $\alpha \in (0, 1)$, $p \succ_{GK} q$ implies $\alpha p + (1 - \alpha) r \succ_{GK} \alpha q + (1 - \alpha) r$).

Proof of claim 2: Suppose that $p \succ_{GK} q$. Let $s \succ \alpha p + (1 - \alpha) r$. We need to show that $s \succ \alpha q + (1 - \alpha) r$. Without loss of generality, we assume $p \neq p^M$.

First, we show that there exists $t \in C$ such that $t \succ p$ and $s \succ \alpha t + (1 - \alpha) r$. Since $s \succ \alpha p + (1 - \alpha) r$, (A.3) implies the existence of $\beta \in (0, 1)$ such that $s \succ (1 - \beta)p^M + \beta(\alpha p + (1 - \alpha) r)$. By mixture monotonicity and (A.3),

$$s \succ (1 - \beta)p^M + \beta(\alpha p + (1 - \alpha) r) \succ \alpha(1 - \beta)p^M + \alpha\beta p + (1 - \alpha)r \quad (7)$$

Define $t := (1 - \beta)p^M + \beta p$. Then $t \succ p$. Also, by the above equation, $s \succ \alpha t + (1 - \alpha) r$.

$p \succ_{GK} q$ and $t \succ p$ together implies $t \succ q$. Then independence axiom, (A.3), implies $s \succ \alpha t + (1 - \alpha) r \succ \alpha q + (1 - \alpha) r$. Therefore, $s \succ \alpha q + (1 - \alpha) r$. ♣

Claim 3. Let p^M and p^m be the greatest and smallest elements of C , respectively. Then, for each $q \in C$, there exist $\underline{\alpha}(q), \bar{\alpha}(q) \in (0, 1)$ such that $\alpha p^M + (1 - \alpha) p^m \succ q$ for all $\alpha > \bar{\alpha}(q)$ and $q \succ \alpha p^M + (1 - \alpha) p^m$ for all $\alpha < \underline{\alpha}(q)$.

Proof of claim 3: Let $S_q^+ = \{\alpha \in [0, 1] \mid \alpha p^M + (1 - \alpha) p^m \succ q\}$. Since S_q^+ is not empty (e.g., $1 \in S_q^+$) and bounded, the infimum of S_q^+ exists. Let $\bar{\alpha}(q) = \inf S_q^+$. By mixture monotonicity, $\alpha > \bar{\alpha}(q)$ implies $\alpha \in S_q^+$.

Next we show that $\bar{\alpha}(q) \notin S_q^+$. Suppose, by way of negation, that $\bar{\alpha}(q) \in S_q^+$ then, by (A.2), there is $\beta \in (0, 1)$ such that $\beta(\bar{\alpha}(q) p^M + (1 - \bar{\alpha}(q)) p^m) + (1 - \beta) p^m \succ q$. Hence $\beta \bar{\alpha}(q) p^M + (1 - \beta \bar{\alpha}(q)) p^m \succ q$. By mixture monotonicity, $\alpha p^M + (1 - \alpha) p^m \succ q$ for all $\alpha > \beta \bar{\alpha}(q)$. But $\bar{\alpha}(q) > \beta \bar{\alpha}(q)$, thus $\bar{\alpha}(q)$ is not a lower bound of S_q^+ . A contradiction.

¹⁵See Kreps (1988).

Let $\underline{\alpha}(q)$ be the supremum of $S_q^- := \{\alpha \in [0, 1] \mid q \succ \alpha p^M + (1 - \alpha) p^m\}$. By similar argument, $\alpha \in S_q$ for all $\underline{\alpha}(q) > \alpha$, and $\underline{\alpha}(q) \notin S_q^-$. ♣

Claim 4. For all $q \in C$, $\bar{\alpha}(q) p^M + (1 - \bar{\alpha}(q)) p^m \succeq q$ and $q \succeq \underline{\alpha}(q) p^M + (1 - \underline{\alpha}(q)) p^m$.

Proof of claim 4: Let $r \succ \bar{\alpha}(q) p^M + (1 - \bar{\alpha}(q)) p^m$ then, by (A.2), there is $\beta \in (0, 1)$ such that $r \succ [\beta(1 - \bar{\alpha}(q)) + \bar{\alpha}(q)] p^M + (1 - \beta(1 - \bar{\alpha}(q)) + \bar{\alpha}(q)) p^m$. But $\bar{\alpha}(q) < \bar{\alpha}(q) + \beta(1 - \bar{\alpha}(q))$, hence

$$[\beta(1 - \bar{\alpha}(q)) + \bar{\alpha}(q)] p^M + (1 - \beta(1 - \bar{\alpha}(q)) + \bar{\alpha}(q)) p^m \in S_q^+.$$

Thus, by transitivity, $r \succ q$. Hence, by Definition 1, $\bar{\alpha}(q) p^M + (1 - \bar{\alpha}(q)) p^m \succ_{GK} q$.

Moreover, $\bar{\alpha}(q) \notin S_q^+$ implies that $\neg(q \succ \bar{\alpha}(q) p^M + (1 - \bar{\alpha}(q)) p^m)$. Hence, by Definition 1, $\bar{\alpha}(q) p^M + (1 - \bar{\alpha}(q)) p^m \succeq q$.

The proof that $q \succeq \underline{\alpha}(q) p^M + (1 - \underline{\alpha}(q)) p^m$ is by the same argument. ♣

For every $p \in C$, let $\mathcal{L}(p)$ be the linear subspace spanned by the vectors $(\bar{\alpha}(p) - p)$ and $(p^M - p^m)$.¹⁶

Claim 5. The functions $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$ are affine on $\mathcal{L}(p) \cap C$.

Proof of claim 5: Let $p, q \in \mathcal{L}(p)$ and suppose that $p \succ_{GK} q$. Define $\varphi(p) = \bar{\alpha}(p) p^M + (1 - \bar{\alpha}(p)) p^m$. Then $\varphi(p) - p$ is parallel to $\varphi(q) - q$. To see this, suppose, by way of negation that $\varphi(p) - p$ is not parallel to $\varphi(q) - q$, then

$$\{q + \lambda(\varphi(p) - p) \mid \lambda \geq 0\} \cap \langle p^m, p^M \rangle = \varphi(q) + \mu(p^M - p^m), \quad (8)$$

where $\langle p^m, p^M \rangle := \{\lambda p^M + (1 - \lambda) p^m \mid \lambda \in \mathbb{R}\}$ is the line that goes through p^M and p^m .

Two vectors being nonparallel implies that $\mu \neq 0$. Without loss of generality, assume that $\mu > 0$. By claim 1 and (A.3), $\beta p + (1 - \beta) q \succ_{GK} q$. For sufficiently small β , $\varphi(q) + \mu(p^M - p^m) \succ \varphi(\beta p + (1 - \beta) q) \succ_{GK} \varphi(q)$. Then there exists $r \in (\varphi(\beta p + (1 - \beta) q), \varphi(q))$. This means $r \succ \beta p + (1 - \beta) q$ and $\neg(r \succ q)$. This contradicts $\beta p + (1 - \beta) q \succ_{GK} q$.

The affinity of $\underline{\alpha}(\cdot)$ is proved similarly. ♣

Claim 6. The function $\bar{\alpha}(\cdot)$ is convex on C .

¹⁶That is, $\mathcal{L}(p) = \{q \in C \mid q = \xi(\bar{\alpha}(p) - p) + \lambda(p^M - p^m), \xi, \lambda \in \mathbb{R}\}$.

Proof of claim 6: Let p and q be such that $\mathcal{L}(p) \neq \mathcal{L}(q)$ and $\bar{\alpha}(p) = \bar{\alpha}(q) = \hat{\alpha}$. Thus, $\hat{\alpha}p^M + (1 - \hat{\alpha})p^m \succ_{GK} p, q$. By Claim 1, $\beta p + (1 - \beta)(\hat{\alpha}p^M + (1 - \hat{\alpha})p^m) \succ_{GK} \beta p + (1 - \beta)q$, for all $\beta \in [0, 1]$. Moreover, since $\hat{\alpha}p^M + (1 - \hat{\alpha})p^m \in \mathcal{L}(p)$, by Claim 5

$$\bar{\alpha}(\beta p + (1 - \beta)(\hat{\alpha}p^M + (1 - \hat{\alpha})p^m)) = \beta\bar{\alpha}(p) + (1 - \beta)\bar{\alpha}(\hat{\alpha}p^M + (1 - \hat{\alpha})p^m) = \hat{\alpha}.$$

Hence, by Definition 1, $\hat{\alpha} \geq \bar{\alpha}(\beta p + (1 - \beta)q)$. Thus, $\bar{\alpha}(\beta p + (1 - \beta)q) \leq \hat{\alpha} = \beta\bar{\alpha}(p) + (1 - \beta)\bar{\alpha}(q)$. ♣

Let $\{e^i \in \mathcal{L} \mid i = 1, \dots, n\}$ be the canonical basis of $\mathcal{L}$. By mixture monotonicity, $p^M = e^i$ and $p^m = e^j$, for some i, j . Without loss of generality, let $p^M = e^n$ and $p^m = e^1$. For every $i \in \{1, \dots, n - 2\}$ and $\lambda \in [0, 1]$, define $q(i, \lambda) = \lambda e^i + (1 - \lambda)e^{i+1}$, and $q(n - 1, \lambda) = \lambda e^{n-1} + (1 - \lambda)e^1$. For every $p \in C$, let $\mathcal{L}(p, q(i, \lambda))$ be the linear subspace spanned by p^M, p and $q(i, \lambda)$.

Let $J = \{p \in C \mid e^M \succ p \succ e^i, i = 1, \dots, n - 1\}$. For every $\lambda \in [0, 1]$, define

$$\bar{\alpha}^{i\lambda}(p) = \inf\{\alpha \in [0, 1] \mid \alpha e^M + (1 - \alpha)q(i, \lambda) \succ p\}.$$

By the same argument as above, $\bar{\alpha}^{i\lambda}(\cdot)$ is a convex function on J whose restriction to $\mathcal{L}(p, q(i, \lambda))$ is affine.

Fix $\bar{p} \in J$ and let $Q = \{q \in J \mid \bar{\alpha}^{i\lambda}(q) = \bar{\alpha}^{i\lambda}(\bar{p})\}$. The every $p \in C$ may be expressed as $p = \varsigma p^M + (1 - \varsigma)q$ for some $q \in Q$ and $1 \geq \varsigma$. Extend $\bar{\alpha}^{i\lambda}(\cdot)$ to C by defining $\bar{\alpha}^{i\lambda}(p) = 1 - (1 - \varsigma)\bar{\alpha}^{i\lambda}(\bar{p})$.

By convexity, $\bar{\alpha}^{i\lambda}(\cdot)$ is differentiable everywhere, except possibly at a countable number of points. For every $p \in C$ at which $\bar{\alpha}^{i\lambda}(\cdot)$ is differentiable, denote by $\nabla \bar{\alpha}^{i\lambda}(p)$ the gradient vector of $\bar{\alpha}^{i\lambda}(\cdot)$ at p .¹⁷ The affinity of $\bar{\alpha}^{i\lambda}(\cdot)$ on $\mathcal{L}(p, q(i, \lambda))$ implies that $\nabla \bar{\alpha}^{i\lambda}(p) = \nabla \bar{\alpha}^{i\lambda}(r)$ for all $r \in \mathcal{L}(p, q(i, \lambda))$.

Define $G = \{\nabla \bar{\alpha}_p^{i\lambda} \in \mathbb{R}^n \mid i = 1, \dots, n - 1, \lambda \in [0, 1], p \in Q\}$. For each $u \in \mathbb{R}^n$ define a function $U : C \rightarrow \mathbb{R}$ by $U(q) = u \cdot q$. Let $\mathcal{U} := \{U \mid u \in G\}$, then, by definition, $U \in \mathcal{U}$ implies that U is affine.

By definition of $\bar{\alpha}^{i\lambda}(\cdot)$, $q \succ p$ implies $\bar{\alpha}^{i\lambda}(q) > \bar{\alpha}^{i\lambda}(p)$, for all $\bar{\alpha}^{i\lambda}(\cdot)$, and $q \succ_{GK} p$ implies $\bar{\alpha}^{i\lambda}(q) \geq \bar{\alpha}^{i\lambda}(p)$, for all $\bar{\alpha}^{i\lambda}(\cdot)$. But $\bar{\alpha}^{i\lambda}(q) > \bar{\alpha}^{i\lambda}(p)$ if and only if $\bar{\alpha}_r^{i\lambda}(q) > \bar{\alpha}_r^{i\lambda}(p)$ for all $i = 1, \dots, n - 1, \lambda \in [0, 1]$ and $r \in Q$. Thus,

¹⁷Denote by $G_{\bar{\alpha}^{i\lambda}}$ the epigraph of $\bar{\alpha}^{i\lambda}(\cdot)$, then $G_{\bar{\alpha}^{i\lambda}}$ and C are convex sets, hence, so is $G_{\bar{\alpha}} \cap C$. For every $p \in C$ such that $e^n \succ p \succ e^i$, for all $i = 1, \dots, n - 1$, and let $H(u_{\bar{\alpha}^{i\lambda}, p}, \bar{\alpha}^{i\lambda}(p))$ be a supporting hyperplane of $G_{\bar{\alpha}^{i\lambda}}$ at p . That such a hyperplane exists follows from the fact that the algebraic interior, $G_{\bar{\alpha}^{i\lambda}}^o$, of $G_{\bar{\alpha}^{i\lambda}}$ is nonempty (e.g., $(\frac{1}{2}\bar{\alpha}(p) + \frac{1}{2}1, p) \in G_{\bar{\alpha}}^o$).

by definition $q \succ p$ implies $U(p) > U(q)$ for all $U \in \mathcal{U}$ and $q \succsim_{GK} p$ implies $U(p) \geq U(q)$.

To show the converse, let $U(q) \geq U(p)$ for all $U \in \mathcal{U}$, and suppose, by way of negation, that not $q \succsim_{GK} p$. If $p \succ q$ then, by necessity, $U(p) > U(q)$, for all $U \in \mathcal{U}$, which is a contradiction. Suppose that $q \bowtie p$ and let $\partial B(r)$ denote the boundary of the upper contour set, $B(r)$, of r .

Claim 7: For all $q, p \in C$, $q \bowtie p$ implies that $\partial B(q) \cap \partial B(p) \neq \emptyset$.

Proof of claim 7: Since $B(p)$ and $B(q)$ are full dimensional cones, $q \bowtie p$ implies that there exist $d \in \partial B(q)$ such that the ray $\langle d, q \rangle := \{\xi(d - q) \mid \xi > 0\}$, intersects $B(p)$. Let $r = \langle d, q \rangle \cap B(p)$. Thus, $r \in \partial B(q) \cap \partial B(p)$. ♣

Choose $r \in \partial B(q) \cap \partial B(p)$, then $r \succsim_{GK} p$ and $r \succsim_{GK} q$. Let $t = \langle p, r \rangle \cap \partial C$ then, by definition, $U(t) = U(p) = U(r)$ for some $U \in \mathcal{U}$. Moreover, $U(t) > U(q)$ implying that $U(r) > U(q)$. Hence, $U(p) > U(q)$. A contradiction.

Hence, $q \succsim_{GK} p$ if and only if $U(q) \geq U(p)$, for all $U \in \mathcal{U}$. By the same argument, $q \succ p$ if and only if $U(q) > U(p)$ for all $U \in \mathcal{U}$. This complete the proof that (i) implies (ii).

The proof the (ii) implies (i) is straightforward.

3.2 Proofs of the uniqueness theorem

Following Shapley and Baucells (1998), without loss of generality, let C be efficiently embedded in $\mathcal{L}$ and suppose that the origin of $\mathcal{L}$ is in C . Fix an interior point ξ of C , and let $B(\xi) := \{\zeta \in C \mid \zeta \succ \xi\}$. Then, $B(\xi)$ is a convex cone with vertex ξ (see Shapley and Baucells (1998) Lemma 1.4). Let $\bar{B}(\xi) := \{\vartheta \in \mathcal{L} \mid \vartheta - \xi = \lambda(\zeta - \xi), \zeta \in B(\xi) \text{ and } \lambda > 0\}$. Next, translate the extended cone, $\bar{B}(\xi)$, to the origin by subtracting ξ . Denote the translated cone B . Note that B is a convex cone in $\mathcal{L}$ with vertex 0, and it does not depend on which interior point of C we started from (see Shapley and Baucells (1998) Lemma 1.3). For all $\zeta \in C$, $B(\zeta) = (\zeta + B) \cap C$. Moreover, for all $\zeta, \xi \in C$, $\zeta \succ \xi$ if and only if $\zeta - \xi \in B$. Thus, by Shapley and Baucells (1998) theorem 1.6, the cone B completely characterizes the preference relation $\succ$.

Note that every vector in $\eta \in B$ has the form $\lambda(\vartheta - \zeta)$, for some $\vartheta, \zeta \in C$ such that $\vartheta \succ \xi$. Suppose that there exist $V \in \langle \mathcal{V} \rangle / \langle \mathcal{U} \rangle$. Since $\langle \mathcal{U} \rangle$ is a convex cone, by the separating hyperplane theorem, there exists ξ in $\mathcal{L}$, $\xi \neq 0$, such that $U \cdot \xi > 0 \geq V \cdot \xi$ for all $U \in \langle \mathcal{U} \rangle$. But the constant vectors θ are in $\langle \mathcal{U} \rangle$.

Hence, $0 \geq \theta \cdot \xi = \theta \sum_{i=1}^n \xi_i$ for all $\theta \in \mathbb{R}$. Thus, $\sum_{i=1}^n \xi_i = 0$. But, by Theorem 1, $U \cdot \xi > 0$ for all $U \in \langle \mathcal{U} \rangle$ implies that $\xi \in B$, hence $\xi = \lambda(\vartheta - \zeta)$, for some $\vartheta, \zeta \in C$ such that $\vartheta \neq 0 \neq \zeta$. Substituting for ξ in the above inequalities we get: $U \cdot \vartheta > U \cdot \zeta$, for all $U \in \mathcal{U}$ and $V \cdot \zeta \geq V \cdot \vartheta$. A contradiction.

The case $U \in \langle \mathcal{U} \rangle / \langle \mathcal{V} \rangle$ is ruled out by the same argument. Hence, $\langle \mathcal{V} \rangle = \langle \mathcal{U} \rangle$. ■

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