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Fair Allocation

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PREFACE

What is fairness? What can mathematics say about it? In the abstract fairness sounds appealing, but can it be pinned down? Without pretending to give a definitive answer to this question, this volume looks at the problem of fair allocation through six examples: the apportionment of representation, the measurement of income inequality, the allocation of joint costs, the levying of taxes, the design of voting rules, and the framing of auction procedures. While these topics by no means exhaust the field, they are representative of it and display its breadth. In each, fairness has a somewhat different significance, yet common axiomatic threads are also apparent that suggest broader principles at work, and that give the subject mathematical unity.

Three considerations motivated the selection of topics. First, each is a real problem. Representation "according to numbers" is mandated in the U.S. Constitution, and similar provisions are found in most representative forms of government. Measures of inequality are used to assess differences between societies in the distribution of income and wealth. Joint cost allocation is practiced by many firms for purposes of accounting and control, and is required for most public utilities. Income taxes are a fact of life. Voting is a widely used approach to finding group consensus. Auctions represent a convenient and accepted method of arbitrating the exchange of property.

Second, each of the above topics is concerned with "norms" or "standards" for evaluating allocations or for effecting them. These include norms of comparative equity such as measures of inequality, and standards of procedure such as voting rules and auctions. It is this focus on normative properties that is susceptible of mathematical analysis. Concepts of fairness suggested by precedent, logic, and common sense can sometimes be captured as formal properties or principles, which can then be used via the axiomatic method to characterize known methods and to discover new ones.

Third, each of the topics selected has received enough mathematical development recently to justify an up-to-date survey. These surveys of the literature are augmented with examples, statements of theorems, and sketches of proofs in order to be useful as teaching devices.

While related in some ways to game theory--indeed game theory provides a useful tool for analyzing many types of allocation problems--the subject of fair allocation differs from game theory in its emphasis on standards rather than on strategy, and on equity rather than rationality.

Other topics might have been included, but for want of space, time, or inclination were not. These include the classical "how to cut a cake" problem of Steinhaus, bargaining theory, risk-sharing, experimental results, and "envy-free" allocations in exchange economies. As stated at the outset, the present volume represents a sampling of problems that are firmly rooted in practice. It is hoped that this collection will provide new teaching materials, suggest novel applications, and stimulate further research.

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