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# Equilibrium selection in bargaining models

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## Abstract

This paper examines evolutionary equilibrium selection in bargaining models. We show that random best-response and continuous best-response learning dynamics give rise to (different) simple sufficient conditions for identifying outcomes as stochastically stable. This allows us to characterize the implications of these dynamics in simple bargaining games.

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## 1. Introduction

For many years, the two-player bargaining problem was held to be indeterminate. The essence of the difficulty is captured by the Nash demand game, where any pair of efficient, individually rational and mutually compatible demands is a strict Nash equilibrium (Nash, 1953). In response, Nash proposed a refinement of the game that eliminated all equilibria except those close to the Nash bargaining solution.

Rubinstein's (1982) and Ståhl (1972) proposed extensive-form bargaining models in which players alternate in making offers to one another. These games have unique subgame-perfect equilibria that lie close to the Nash bargaining solution under suitable conditions. This solution initially appeared to close the issue, but concerns about the viability of equilibrium refinements has reopened the equilibrium selection question.

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Bob Rosenthal was an active contributor to the game-theoretic literature on bargaining (Fernandez and Rosenthal, 1990; Lang and Rosenthal, 2001; Rosenthal, 1978, 1985, 1993) as well as equilibrium selection (Brown and Rosenthal, 1990; Glazer and Rosenthal, 1992; Rosenthal, 1981). His Centipede Game became a standard example illustrating the fragility of subgame-perfect equilibrium and the backward induction principle on which it rests. One outcome of the equilibrium-refinements literature was to direct attention to the trial-and-error adaptive processes by which people might find their way towards equilibrium. Again, we find Bob Rosenthal contributing to the discussion (Friedman and Rosenthal, 1986; Gale and Rosenthal, 1999, 2001; Rosenthal, 1989, 1993a).

In writing for an issue in Bob's honor, we therefore thought it appropriate to examine applications of evolutionary equilibrium selection to the bargaining problem. We have three goals. The first is to explore conditions under which it is easy to identify the equilibria that will be selected in adaptive models with persistent perturbations (Foster and Young, 1990; Kandori et al.'s, 1993, and Young 1993, 1993a, 1998). The second is to study the implications for equilibrium selection of alternative specifications of the best-response adjustment dynamics. The third is to show how our framework relates to other models of the two-player bargaining problem and to provide some links between results in the literature.

## 2. Evolutionary equilibrium selection

Suppose that a finite normal-form game  $G$  is to be played repeatedly by players selected from one or more large but finite populations of equal size. Our applications in this paper are two-player bargaining games in which player 1 is always drawn from one population of size  $N$  and player 2 from another population of size  $N$ , but the framework outlined in this section applies to Markov processes quite generally.

In each time period, the current state of the process is the proportion of each population playing each of the pure strategies in  $G$ . Over time the state evolves as agents adjust their strategies in response to the behavior of the other players. If the evolutionary dynamics are deterministic and the game  $G$  has multiple equilibria, the equilibrium selected will usually depend on initial play. For example, suppose that each agent selected to play the Nash demand game in a given period chooses a best response to previous-period play. Any division giving both players a positive share of the surplus is a limiting outcome of the resulting dynamics if players are already demanding something sufficiently close to that division in the initial state.

Suppose instead that agents choose best responses with high probability, but otherwise choose strategies at random. These random choices, called *mutations*, can arise for many reasons, including trembles, experimentation, idiosyncratic shocks to utility functions, or mistaken perceptions of the environment. The resulting stochastic dynamical system can differ fundamentally from the associated deterministic system even when the magnitude of the perturbations is arbitrarily small. The mutations occasionally bounce the process from one equilibrium to another, giving rise to an ergodic Markov chain that has a unique long-run probability of being in any given state, independently of initial conditions. This is the *stationary distribution* of the process. When the mutation probabilities are sufficiently small, probability is typically attached to only a small class of states, called

the *stochastically stable states* (Foster and Young, 1990). Under conditions that we shall discuss below, there is often a unique stochastically stable population state that corresponds to a particular Nash equilibrium of the game. We say that this equilibrium is selected in the long (or ultralong) run.

### 2.1. Analysis of Markov chains

The conventional analysis of Markov chains involves solving a system of linear equations for the stationary distribution. For the problems we are interested in, this algebraic method is cumbersome. Finding the stationary distribution of a  $2 \times 2$  game with a population of 100 players in each role would require solving  $101^2 = 10,201$  linear equations for the probabilities attached to the 10,201 states.

A more tractable approach is available using directed graphs. This geometric method was pioneered by Freidlin and Wentzell (1984) for continuous-time, continuous-state processes and was subsequently adapted by Young (1993) and Kandori et al.'s (1993) for finite chains with discrete perturbations. It has become the standard method for characterizing the long-run behavior of evolutionary models with persistent stochastic perturbations.<sup>1</sup>

Consider a time-homogeneous Markov chain defined on a finite state space  $X$ . Identify each state  $x \in X$  by a node of a graph. A tree rooted at  $x$  consists of  $|X| - 1$  directed edges such that from every node  $y \neq x$  there exists one and only one directed path from  $y$  to  $x$ . The idea is illustrated in Fig. 1. Let  $p_{xy}$  be the one-step transition probability from state  $x$  to state  $y$ . Given a rooted tree  $T$ , let  $P(T)$  denote the product of the probabilities  $p_{xy}$  over all directed edges  $(x, y)$  in the tree. Let  $\nu(x) = \sum P(T)$ , where the sum is taken over all trees rooted at  $x$ . The *Markov chain tree theorem* states that for any irreducible finite chain,  $\nu(x)$  is proportional to the long-run probability of state  $x$  (Freidlin and Wentzell, 1984, Lemma 3.1, p. 177). In other words, the normalized values  $\nu(x)/\sum_y \nu(y)$  yield the unique stationary distribution of the process.

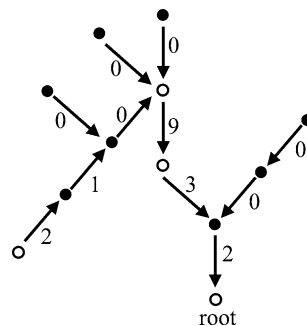


Fig. 1. Illustration of a tree, where numbers are one-step resistances  $r(x, y)$  (Section 2.3) and hollow nodes denote equilibria.

<sup>1</sup> Examples include, among many others, Binmore and Samuelson (2002), Ellison (1993, 2000), Nöldeke and Samuelson (1993), Samuelson (1994, 1997), and Young (1993a, 1998, 1998a).

## 2.2. Learning dynamics and stochastic stability

We now specialize to the situation where states are distributions of strategies in a population and the Markov process is a perturbed version of best-response dynamics. Let  $x_t$  denote the *state* of the system at time  $t$ , i.e., a vector identifying, for each population (of size  $N$ ) and each pure strategy, the proportion of the agents in the population playing that strategy.<sup>2</sup>

Let  $\Gamma_{(0,N)}$  be a Markov process defined on the state space  $X$ . We interpret  $\Gamma_{(0,N)}$  as the evolutionary process by which agents adapt their strategies to the behavior of the other agents in the game. In an economic context,  $\Gamma_{(0,N)}$  might represent a process of learning or cultural evolution. In a biological context,  $\Gamma_{(0,N)}$  represents the forces of reproduction.

At the beginning of each period  $t$ , a new state is drawn according to the process  $\Gamma_{(0,N)}$ , given the current state  $x_t$ . With probability  $\epsilon$ , each agent is then independently drawn to be a “mutant.” Mutants independently select a new strategy at random from the relevant pure strategy set, with positive probability attached to each strategy. The result is the state  $x_{t+1}$  with which the system enters period  $t + 1$ . Models of this type were pioneered by Kandori et al.’s (1993) and Young (1998).

The combination of the process  $\Gamma_{(0,N)}$  and mutations induces a Markov process denoted by  $\Gamma_{(\epsilon,N)}$ . The mutations ensure that the transition matrix for  $\Gamma_{(\epsilon,N)}$  is strictly positive. As a result,  $\Gamma_{(\epsilon,N)}$  has a unique stationary distribution that is the limiting distribution for the period- $t$  state as  $t$  gets large and is the limiting distribution for the proportion of time spent in each state along almost all sample paths.<sup>3</sup> Let  $\mu_{(\epsilon,N)}^*$  denote this stationary distribution. We will be interested in the limit of the stationary distributions as the probability of a mutation approaches zero, or

$$\lim_{\epsilon \rightarrow 0} \mu_{(\epsilon,N)}^* \tag{1}$$

We say that a state is *stochastically stable* if it receives positive probability in this limit (Young, 1993).

## 2.3. Identifying stochastically stable states

For a fixed value of  $\epsilon$ , the Markov chain tree theorem tells us that states will receive large probabilities in the stationary distribution if they are the roots of trees whose transition probabilities are relatively large. When  $\epsilon$  is arbitrarily small, transitions involving more mutations will become arbitrarily less likely than transitions involving fewer mutations. In the limit as  $\epsilon$  approaches zero, the total number of mutations in a tree will swamp all other considerations. A state will then be stochastically stable if and only if there is no other state with a tree involving fewer mutations.

Formally, given any two states  $x, y \in X$ , define the *one-step resistance* to be the minimum number of mutations needed for  $\Gamma_{(\epsilon,N)}$  to move from  $x$  to  $y$  with positive

<sup>2</sup> We use “players” to refer to the underlying game and “agents” to refer to the members of the associated populations.

<sup>3</sup> For example, Hoel et al. (1972, Chapter 2, Theorem 5, Corollary 6, and Theorem 7).

probability in one step. Define the *resistance* of a rooted tree to be the sum of the one-step resistances associated with its directed edges. A *minimal tree* is a rooted tree with minimum resistance. It follows from the Markov chain tree theorem that *a state  $x$  is stochastically stable if and only if it is the root of a minimal tree.*

Concentrating on the limit as the mutation probability  $\epsilon$  gets arbitrarily small thus simplifies the application of the Markov chain tree theorem, allowing us to simply count mutations rather than deal with products of transition probabilities. The process of finding a minimal tree can often be further simplified by thinning out the number and types of trees that need to be considered.

First, a major simplification is possible if the following no-cycling condition holds:

**No-cycling condition.** The mutation-free process  $\Gamma_{(0,N)}$  satisfies the *no-cycling condition* if all of its absorbing sets are singletons.

When players update asynchronously to best responses, the no-cycling condition holds for a wide class of games known as acyclic games (Young, 1993), which includes coordination games and bargaining games as special cases.

If the no-cycling condition holds and the process without mutations is a best-response dynamic, then it converges with probability one to a Nash equilibrium state from any starting point. To see why, recall that every finite time-homogeneous process reaches an absorbing set with probability one. The no-cycling property implies that this set consists of a single absorbing state, which must be a Nash equilibrium under the best-response dynamics. The no-cycling condition thus simplifies the search for a minimal tree by allowing us to consider only trees whose roots are absorbing states (and hence Nash equilibria).

Much more than this is true. Define a *least-resistant path* from state  $x$  to state  $y$  as a sequence of one-step transitions that begins at  $x$  and ends at  $y$  and whose total resistance is as small as possible. Then it is straightforward to verify:

**Lemma 1** (Pruning property). *Given the no-cycling condition, all of the positive resistances in every minimal rooted tree arise along least-resistant paths between absorbing states.*

Hence all states that do not lie on such least-resistant paths can be pruned away when searching for a minimal tree. The pruning operation is illustrated in Fig. 2. This smaller tree is obtained from the tree in Fig. 1 by eliminating all nodes that do not lie on the minimal paths that connect the four equilibria.

Let  $X^*$  be the set of absorbing states. This is a subset of all Nash equilibrium states under best-response dynamics, and may be a strict subset, since in some best-response dynamics mixed equilibria will not be absorbing. For each pair  $x^*, y^* \in X^*$  let  $r^*(x^*, y^*)$  denote the minimum number of mutations needed to move from  $x^*$  to  $y^*$  in a series of one-step transitions. A *miniature (Bonsai) tree* is a rooted tree whose nodes are the states in  $X^*$  with each directed edge  $x^* \rightarrow y^*$  given weight  $r^*(x^*, y^*)$ .

We can identify the roots of minimum trees in the original state space  $X$  by considering just the Bonsai trees (Samuelson, 1994; Young, 1993, Theorem 4):

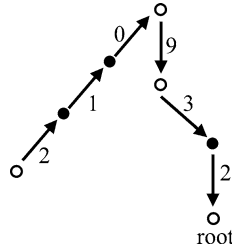


Fig. 2. Pruned version of the tree of Fig. 1.

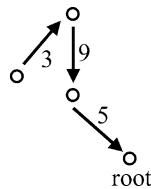


Fig. 3. Bonsai version of the pruned tree of Fig. 2, where numbers are the values  $r^*(x^*, y^*)$ .

**Lemma 2** (Bonsai property). *If the no-cycling condition holds, then the roots of the minimum Bonsai trees are the same as the roots of the minimum trees defined over the entire state space  $X$ .*

In other words, once the resistances between equilibria have been calculated, the geometry of the least resistant paths is irrelevant. Figure 3 shows the Bonsai tree corresponding to the pruned tree in Fig. 2.

The search for a minimal tree thus reduces to the search for a minimal Bonsai tree (given the no-cycling condition). We now introduce two conditions that simplify the problem of identifying the minimum Bonsai tree.

**Proposition 1** (Local resistance test). *Let the no-cycling condition hold.*

- (1.1) *Let  $x$  be an absorbing state such that the largest incoming resistance  $r^*(y, x)$  is less than the smallest outgoing resistance  $r^*(x, z)$ . Then  $x$  is the root of every minimal tree.*
- (1.2) *Let  $x$  be an absorbing state such that the smallest incoming resistance  $r^*(y, x)$  is larger than the smallest outgoing resistance  $r^*(x, z)$ . Then  $x$  is not the root of any minimal tree.*

**Proof.** Let  $x$  satisfy condition (1.1) and suppose that some minimal tree has root  $y \neq x$ . Then we can create a new tree with root  $x$  by introducing the edge  $y \rightarrow x$  and deleting the edge which begins the path that leads from  $x$  to  $y$ . Because the new edge has smaller resistance than the deleted edge, the new tree has smaller total resistance, a contradiction.

Let  $x$  satisfy condition (1.2) and suppose that some minimal tree has root  $x$ . Let  $y$  be a node such that the resistance  $x \rightarrow y$  is minimized among all outgoing edges from  $x$ . Adding edge  $x \rightarrow y$  results in a unique cycle that contains  $x$ . Let  $z \rightarrow x$  be the unique edge of the cycle directed toward  $x$ . By assumption the resistance on  $z \rightarrow x$  is larger than on  $x \rightarrow y$ . Hence, by deleting  $z \rightarrow x$  we obtain a tree rooted at  $z$  that has smaller total resistance than the original tree, a contradiction.  $\square$

A particular instance of the local-resistance test arises when the number of mutations required to jump from the equilibrium  $x$  to any other equilibrium involves more than half the total number of agents, while the number of mutations needed to go in the other direction involves less than half the total number of agents. Abusing existing usage slightly, we say that such an equilibrium is *half dominant*.<sup>4</sup> An immediate consequence of the local-resistance test is then (Ellison, 2000, Corollary 1):

**Corollary 1.** *Every half-dominant equilibrium is stochastically stable.*

When the local resistance test fails to identify the minimum tree, there is a simple nonlocal test that often works.

**Proposition 2** (Naive minimization test). *Let the no-cycling condition hold. Join each absorbing state  $x$  to an absorbing state  $y$  that has minimum outgoing resistance  $r^*(x, y)$  from  $x$ . The result is a directed graph with at least one cycle. If the cycle is unique and contains an edge with maximal resistance over all edges, then a minimal tree, called the naive minimization tree, is obtained by deleting this edge.*

**Proof.** Let  $T$  be the naive-minimization tree, with root  $x$ , and let  $T'$  be a tree with root  $y \neq x$ . Since  $x$  is not the root of  $T'$ ,  $T'$  has an edge leading away from  $x$  that is not in  $T$ , say  $x \rightarrow z$ . Similarly  $T$  has an edge leading away from  $y$  that is not in  $T'$ , say  $y \rightarrow w$ . The first of these edges has greater resistance than the second, because the edge emanating from  $x$  that was removed to construct  $T$  has the maximal resistance among all edges from which  $T$  was constructed. Further, every other edge in  $T'$  must have larger resistance than the corresponding outgoing edge in  $T$ , because  $T$  was constructed from the minimal outgoing edges. Hence  $T'$  has greater total resistance than does  $T$ .  $\square$

Both the naive minimization test and local resistance test apply to the coordination examples studied in Binmore and Samuelson (2002). The naive minimization test is also satisfied in Young's analysis of the Nash demand game (Young, 1993a).

Our next condition identifies situations in which it is particularly straightforward to apply the local resistance and naive minimization tests.

<sup>4</sup> In  $2 \times 2$  symmetric games with a single population and either of the best-response dynamics we consider, an equilibrium is half-dominant if and only if it is risk dominant (Harsanyi and Selten, 1988).

**Bandwagon property.** The process  $\Gamma_{(\epsilon, N)}$  satisfies the *bandwagon property* if the minimum number of mutations needed to switch from equilibrium  $x$  to equilibrium  $y$  involves only mutations to strategies that are in  $y$  and not in  $x$ .

Kandori and Rob’s (1998) marginal bandwagon property provides a sufficient condition on the payoffs of the underlying game  $G$  for the bandwagon property to hold. Letting  $(s'_1, s'_2)$  be an equilibrium of  $G$  and  $u_1$  be player 1’s payoff function, the condition is that

$$u_1(s'_1, s'_2) - u_1(s_1, s'_2) \geq u_1(s'_1, s_2) - u_1(s_1, s_2),$$

for any  $s_1$  and  $s_2$ , with an analogous condition for player 2. The interpretation is that deviations from an equilibrium strategy are most costly when the opponent plays her part of the equilibrium. Similar conditions appear in Binmore and Samuelson (2002) and Levine and Pesendorfer (2002).<sup>5</sup>

### 3. Best-response dynamics

This section introduces two variants of the best-response dynamics in  $2 \times 2$  games. Figure 4 shows the space of population states for a  $2 \times 2$  game. Let  $(\sigma_1, \sigma_2)$  denote the pair of mixed strategies in which player 1 attaches probability  $\sigma_1$  to strategy  $s_1$  and player 2 attaches probability  $\sigma_2$  to  $s_2$ . Unless otherwise noted, we assume throughout that  $(s_1, s_2)$  and  $(s'_1, s'_2)$  are strict Nash equilibria. The pure Nash equilibria  $(s_1, s_2)$  and  $(s'_1, s'_2)$  correspond to the states  $(1, 1)$  and  $(0, 0)$ . There is also a mixed Nash equilibrium corresponding to the state  $(\hat{\sigma}_1, \hat{\sigma}_2)$ .

Figure 4 is split into four quadrants by horizontal and vertical lines drawn through the mixed equilibrium  $(\hat{\sigma}_1, \hat{\sigma}_2)$ . When the strategy profile is in the bottom-left quadrant, the best-response vector lies at  $(0, 0)$ , corresponding to  $(s'_1, s'_2)$ . When the strategy profile is in the top-right quadrant, the best response lies at  $(1, 1)$ , corresponding to  $(s_1, s_2)$ . The best

<sup>5</sup> We offer an example similar to one appearing in Young (1993) to show how this condition can fail. Consider the following game:

	$s_2$	$s'_2$	$s''_2$
$s_1$	1	0	0
$s'_1$	0	1	0
$s''_1$	-9	0	0

Consider a state in which all agents in population 1 play  $s_1$  and agents in population 2 play  $s_2$ . Then mutations must switch half of either population  $i$  to strategy  $s'_i$  in order to make strategy  $s'_j$  a best response in the other population, but must shift only a tenth of population  $i$  to  $s''_i$  in order to make  $s'_j$  a best response. Hence, under the random best-response dynamics discussed in more detail below, the least number of mutations required to shift the population from a state corresponding to equilibrium  $(s_1, s_2)$  to a state corresponding to  $(s'_1, s'_2)$  is a tenth of one population, accomplished by having these mutants play strategy  $s''_i$ .

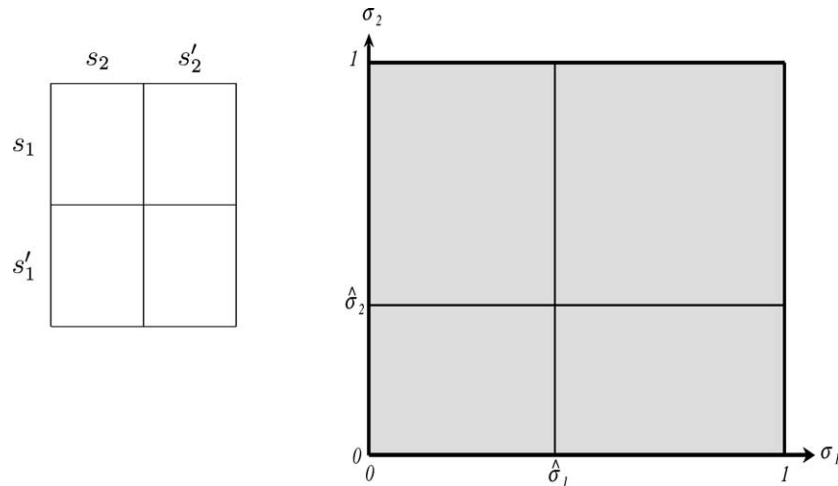


Fig. 4. A  $2 \times 2$  game and corresponding set of population states, where  $\sigma_i$  is the probability attached to strategy  $s_i$ .

response to a state in the bottom-right quadrant is  $(0, 1)$ , or  $(s'_1, s_2)$ , while the best response to a state in the top-left quadrant is  $(1, 0)$ , or  $(s_1, s'_2)$ .

### 3.1. Continuous best-response dynamics

Let the best-response vector  $B(\sigma)$  assign a positive probability to a strategy only if it is a best response to the mixed strategy corresponding to  $\sigma$ . The continuous best-response dynamics is defined on the space of mixed strategies by<sup>6</sup>

$$\frac{d\sigma_t}{dt} = B(\sigma_t) - \sigma_t. \quad (2)$$

The continuous best-response dynamics move from any point in the set of mixed strategies along a straight line towards the current best-response vector. Figure 5 shows some sample trajectories of the best-response dynamics. The set of strategies from which the continuous best-response dynamics lead to a particular equilibrium is called the *basin of attraction* of that equilibrium. In Fig. 5, the dashed line separates the basins of attraction of the pure equilibria,  $(0, 0)$  and  $(1, 1)$ . The dashed line is itself the basin of attraction of the mixed equilibrium.

We now define an analogue of these dynamics for large, finite populations. Let  $L(x)$  be the limiting outcome of state  $x$  under the best-response dynamics given by (2). We assume that the learning dynamics are defined by

$$x_t \rightarrow L(x_t). \quad (3)$$

<sup>6</sup> See Binmore (1992, Chapter 9) for various updating models that lead to this dynamics. Uniqueness typically fails for some values of  $\sigma$ , such as those corresponding to the lines dividing the quadrants in Fig. 4. Formally, (2) is thus a differential inclusion rather than differential equation, which we specialize to a differential equation, as illustrated in Fig. 5.

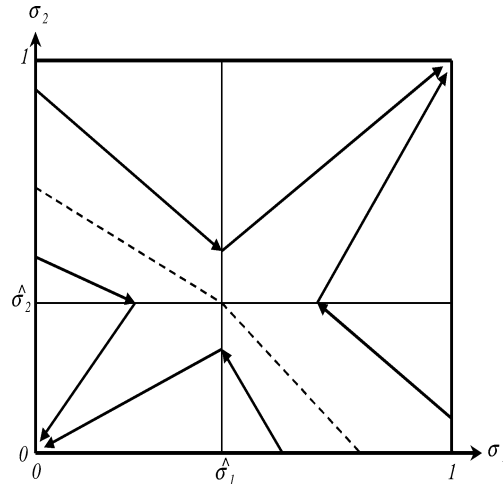


Fig. 5. Basins of attraction and sample trajectories of the continuous best-response dynamics in the game shown in Fig. 4. The dashed line, separating the basins of attraction of  $(0, 0)$  (corresponding to equilibrium  $(s'_1, s'_2)$ ) and  $(1, 1)$  (corresponding to equilibrium  $(s_1, s_2)$ ) is given by  $\sigma_2(1 - \hat{\sigma}_1) + \sigma_1\hat{\sigma}_2 = \hat{\sigma}_2$  (upper-left segment) and  $\hat{\sigma}_1\sigma_2 + (1 - \hat{\sigma}_2)\sigma_1 = \hat{\sigma}_1$  (lower-right segment).

Hence, a state  $x$  is transformed to state  $L(x)$  with probability one.<sup>7</sup> Notice that this formulation contrasts with one in which every agent switches to a best response to the previous-period strategies (as in Kandori et al.'s (1993) *single-population* model), which would allow the two populations to cycle endlessly between the two non-equilibrium states  $(0, 1)$  and  $(1, 0)$ .

In the  $2 \times 2$  game of Fig. 4, let  $R_{01}$  be the proportion of the agents in the two populations that need to mutate from their second pure strategy to their first in order to move the state from the equilibrium  $(0, 0)$  into the basin of attraction of the equilibrium  $(1, 1)$ . This reduces to finding the minimum of  $\frac{1}{2}(\sigma_1 + \sigma_2)$  for  $(\sigma_1, \sigma_2)$  in the basin of attraction of  $(1, 1)$ , and so (ignoring the integer considerations imposed by the finite populations):

$$R_{01} = \begin{cases} \frac{1}{2}(\hat{\sigma}_1 + \hat{\sigma}_2), & \text{if } \hat{\sigma}_1 + \hat{\sigma}_2 \geq 1, \\ \frac{1}{2} \min\{\hat{\sigma}_2/(1 - \hat{\sigma}_1), \hat{\sigma}_1/(1 - \hat{\sigma}_2)\}, & \text{if } \hat{\sigma}_1 + \hat{\sigma}_2 \leq 1. \end{cases} \quad (4)$$

Similarly, we can let  $R_{10}$  be the proportion of agents that need to mutate from their first pure strategy to their second in order to move the state from the equilibrium  $(1, 1)$  into the basin of attraction of the equilibrium  $(0, 0)$ . We can obtain  $R_{10}$  by replacing  $\hat{\sigma}_1$  and  $\hat{\sigma}_2$  by  $1 - \hat{\sigma}_1$  and  $1 - \hat{\sigma}_2$  in (4):

$$R_{10} = \begin{cases} \frac{1}{2}(2 - \hat{\sigma}_1 - \hat{\sigma}_2), & \text{if } \hat{\sigma}_1 + \hat{\sigma}_2 \leq 1, \\ \frac{1}{2} \min\{(1 - \hat{\sigma}_2)/\hat{\sigma}_1, (1 - \hat{\sigma}_1)/\hat{\sigma}_2\}, & \text{if } \hat{\sigma}_1 + \hat{\sigma}_2 \geq 1. \end{cases} \quad (5)$$

<sup>7</sup> We move freely here between the space of mixed strategy profiles and the space of population states. This poses no difficulties as long as we bear in mind that the set of population states is a large but finite collection of points embedded in the space of mixed strategy profiles.

These calculations allow us to conclude:

**Proposition 3.** *Under the continuous best-response dynamics, one of the equilibria in a  $2 \times 2$  game is half dominant:*

$$\begin{aligned} R_{01} &< \frac{1}{2} < R_{10} && \text{when } \hat{\sigma}_1 + \hat{\sigma}_2 < 1, \\ R_{01} &> \frac{1}{2} > R_{10} && \text{when } \hat{\sigma}_1 + \hat{\sigma}_2 > 1. \end{aligned}$$

The local resistance test (Corollary 1) then allows us to conclude that the half-dominant equilibrium is stochastically stable.

### 3.2. Random best-response dynamics

As the name suggests, the random best-response dynamics will be stochastic, in contrast to the deterministic specification of the continuous best-response dynamics. Assume that, in each period, every agent takes an independent draw from a Bernoulli random variable that causes the agent to revise his strategy with probability  $\lambda$ , and has no effect with probability  $1 - \lambda$ . An agent revising his strategy chooses a best response to the current state.

In the case of the random best-response dynamics, the *basin of potential attraction* of an equilibrium is the set of states from which the system moves to the equilibrium with positive probability. In Fig. 4, the basin of potential attraction of each pure equilibrium consists of the three quadrants that do not contain the other pure equilibrium. Notice that basins of (potential) attraction under the random best-response dynamics need not be disjoint, unlike the continuous best-response dynamics.

We can again calculate the proportion of mutations required to move the populations from one equilibrium to the basin of attraction of the other equilibrium in a  $2 \times 2$  game. Ignoring integer considerations, we have:

$$R_{01} = \frac{1}{2} \min\{\hat{\sigma}_1, \hat{\sigma}_2\}, \quad R_{10} = \frac{1}{2} \min\{1 - \hat{\sigma}_1, 1 - \hat{\sigma}_2\}. \quad (6)$$

To leave equilibrium  $(0, 0)$  for the basin of attraction of  $(1, 1)$  in Fig. 6 requires that half of the agents in a single population mutate under the random best-response dynamics, along

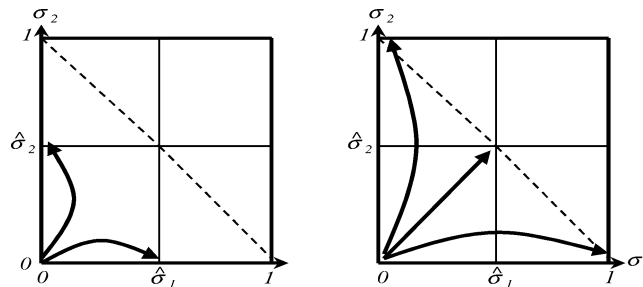


Fig. 6. Comparison of proportion of mutants needed to leave equilibrium  $(0, 0)$  under the random (left) and continuous (right) best-response dynamics.

either of the paths shown in the left panel. Under the continuous best-response dynamics, either half of the agents in *both* populations or *all* of the agents in a single population must mutate, along any of the paths shown in the right panel. Notice also that each equilibrium in Fig. 6 can be escaped under random best-response dynamics with less than half of all agents becoming mutants. Hence:

**Proposition 4.** *Under the random best-response dynamics, neither equilibrium in a  $2 \times 2$  game need be half-dominant.*

As a result, our analysis of the random best-response dynamics will be based on the naive minimization test.

### 3.3. Which dynamics?

In the symmetric  $2 \times 2$  single-population games in which the concept of stochastic stability was first popularized, the random best-response dynamics and continuous best-response dynamics have identical (and disjoint) basins of attraction. Figure 6 shows that this equivalence does not generalize to a two-population model.

We view these dynamics as emerging from the same unperturbed stochastic process, but capturing different ways of introducing mutations and approximating the outcome of this process that will be applicable in different settings. Consider the random best-response dynamics in Fig. 6, with each agent independently drawn to be a mutant with probability  $\epsilon$  in each period. Only half of the agents in population 1 need mutate to strategy  $s_1$  in order to move from a state corresponding to equilibrium  $(s'_1, s'_2)$  to a state in the basin of attraction of equilibrium  $(s_1, s_2)$ . Once the latter state is reached, however, moving to the equilibrium  $(s_1, s_2)$  requires that only agents in population 2 receive opportunities to revise their strategies until at least half of those in population 2 have done so. If the population is very large, this concentration of randomly-drawn strategy-revision opportunities in population 2 is an extraordinarily unlikely event.

The relative likelihoods of various learning sequences are immaterial if, like Young (1993) and Kandori et al.'s (1993), we are concerned with the limit

$$\lim_{\epsilon \rightarrow 0} \mu_{(\epsilon, N)}^* \quad (7)$$

As the mutation rate becomes arbitrarily small, any feasible learning pattern, no matter how improbable, is much more likely than a transition requiring mutations. The key to stochastic stability is then to economize on mutations, no matter what contortions in the learning process are required to avoid a mutation.

Typically, taking the population size to infinity has no effect on the results once we have examined the limiting case of an arbitrarily small mutation rate, and so the limit in (7) can

also be expressed as<sup>8</sup>

$$\lim_{N \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \mu_{(\epsilon, N)}^* \quad (8)$$

As the population size grows, it becomes more and more likely that learn draws will arrive at about the same rate across populations and across the proportions of agents playing each strategy within a population. In the limit as the population becomes arbitrarily large, the outcome of the learning process will lie close to the path of the continuous best-response dynamics with high probability for long periods of time. If we adopt the continuous best-response dynamics as our model of the learning process and then examine the limit as the mutation rate approaches zero, we are thus approximating the following limit:

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \mu_{(\epsilon, N)}^* \quad (9)$$

The first limiting operation ensures that the continuous best-response dynamics provide a good approximation of the learning process.<sup>9</sup>

In addition, we alter the specification of the mutation process, yielding a model of “infrequent mutations” commonly found in biological applications. We assume that in each period there is a probability  $1 - \lambda$  that there are no mutations and probability  $\lambda$  that some mutations occur, with such a positive draw then giving rise to each agent’s being independently selected with probability  $\epsilon$  to be a mutant. We then examine the limit as  $\lambda$  and  $\epsilon$  approach zero, with  $\lambda$  approaching zero more quickly, so that  $\lambda/\epsilon \rightarrow 0$ . The result is that periods featuring mutations become arbitrarily infrequent, even though each such period may feature multiple mutations. In effect, two mutations in a single period become arbitrarily more likely than single mutations in each of two adjacent periods. The biological interpretation of such a model is that multiple mutations are most likely to arise when a single mutation establishes itself in a subgroup of the population, temporarily protected by inbred interactions arising out of kinship relationships or physical isolation, before being thrust into competition with the population at large. A larger mutant bridgehead is less likely to have sufficient time to develop than is a smaller one, captured in our model by the distribution of mutations induced by  $\epsilon$  that is applicable once a burst of mutants appears.<sup>10</sup> Patterns of inbred social interactions may give rise to similar considerations in evolutionary

<sup>8</sup> As  $N$  grows, the state space  $X$  changes. To construct a well-defined limit as  $N$  gets large, we must embed the discrete state space of the Markov process in the continuous strategy space of the game and then examine the weak convergence of the probability measures induced in the latter by the Markov process.

<sup>9</sup> The technical tools for making precise the statement that the learning process will lie close to the continuous best-response dynamics in large populations, and hence for making precise the sense in which the continuous best-response dynamics allow us to examine the limit given in (9), begin with the results of Binmore and Samuelson (1999) and Sandholm (2002), who examine conditions under which stochastic processes with large populations are well approximated by deterministic dynamics.

<sup>10</sup> Our assumption that the  $\epsilon$ -probability mutation draws are independent across agents is not a perfect match for intuition based on isolated subsets of the population. Because the relevant mutant invasions in our analysis will always involve agents playing the same strategy, we could replace this with a more realistic model without affecting the results.

processes based on learning rather than reproduction. We believe that large populations and infrequent mutations are both relevant in many evolutionary applications, prompting us to examine the conjunction of the two.

As periods featuring mutations become arbitrarily infrequent, the learning process proceeds arbitrarily quickly compared to the rate at which mutations appear. We can then conveniently capture the unperturbed continuous best-response dynamics by assuming that we simply move from a state to its limit (assuming the latter exists), leading to (3). The least cost transitions between absorbing states will involve a single burst of mutations, followed by learning according to the continuous best-response dynamics. Which states are stochastically stable will then depend upon the relative sizes of the mutant bursts required to move between absorbing states. We can then ignore the probability  $\lambda$  and apply the techniques of Section 2 to a model in which the learning dynamics are given by (3) and mutations occur with probability  $\epsilon$ , independently across agents and time.

Analyses based on stochastic stability thus potentially involve two limits, as the population size approaches infinity and the probability of a mutation approaches zero. As the probability of a mutation shrinks, the relative probabilities of various transitions involving mutations will be altered. Similarly, as the population size grows, the relative probabilities of various transitions involving the learning process will be altered. When working with (8) and mutation probabilities that are independent across individuals and time, the properties of the invariant distribution will be determined by the probabilities of mutation-driven transitions, with the relative probabilities within the learning process playing no role beyond identifying which transitions are possible. When working with (9) and a mutation process that tends to clump mutations together, relative probabilities of learning transitions will be important, with the paths followed by the continuous best-response dynamics becoming so much more likely than other learning paths as to allow us to neglect the latter.<sup>11</sup>

One can get a sense of the difference between these two limits by comparing the results of Nöldeke and Samuelson (1993) with those of Hart (2001) and Kuzmics (2002). Nöldeke and Samuelson show that stochastic stability need not uniquely select the backward induction outcome in generic extensive-form games of perfect information. In doing so, they examine the limit as the mutation rate goes to zero for a fixed population size, hence effectively working with (8). Their results depend upon the possibility that agents in one population may learn much more rapidly than agents in another population. Hart (2001) examines a case in which the population size must increase sufficiently rapidly as the mutation rate goes to zero, effectively corresponding to (9) (but retaining the independence of mutations across individuals and time). This “smooths” relative rates of learning and mutations across populations, allowing one to show that taking the mutation rate to zero

<sup>11</sup> Both the interchange of the order of limits and the re-specification of the mutation process have substantive effects. In our model, moving from state  $(0, 0)$  (corresponding to equilibrium  $(s'_1, s'_2)$ ) to state  $(1, 1)$  (equilibrium  $(s_1, s_2)$ ) in Fig. 6, while minimizing mutation costs, requires switching all of population 1 to strategy  $s_1$ . If we switched to continuous best-response dynamics but left the mutation process unaltered, the mutation cost of this transition could be reduced by first switching  $\hat{\sigma}_1$  of population 1 to strategy  $s_1$ , and then relying upon a sequence of learning and additional mutations (to  $s_1$ ) to ascend the vertical line at  $\hat{\sigma}_1$  until reaching the basin of attraction of  $(1, 1)$ . The analysis of Section 4 would not apply to this intermediate case.

then uniquely selects the backward induction equilibrium.<sup>12</sup> For example, the unique subgame-perfect equilibrium would be selected in finite versions of Rubinstein's (1982) alternating-offers bargaining game.

Which type of dynamics should we use? This question recalls the debates to which Rosenthal contributed on bargaining models and refinement concepts. Our answer is that such questions cannot be answered independently of the context in which a model is to be applied.

#### 4. Continuous best response in bargaining

The local resistance test and naive minimization test sometimes make it quite easy to do stochastic stability calculations. We demonstrate this by examining some simple bargaining models, beginning in this section with the continuous best-response dynamics and continuing in the next section with the random best-response dynamics. Each of our applications satisfies the no-cycling condition and the bandwagon property.

##### 4.1. Nash demand game

Let  $S$  be a compact, convex subset of  $\mathbb{R}_+^2$  containing the origin. We interpret  $S$  as identifying the set of feasible pairs of utilities upon which players 1 and 2 might agree, with the origin identifying the realized allocation in the event of no agreement. We assume there is a decreasing, differentiable function  $f$  such that  $(s_1, f(s_1))$  is the efficient outcome consistent with any demand  $s_1$  on the part of player 1.

The *Nash demand game* (Nash, 1953) calls for each player to demand a utility, with an allocation being selected in which each player receives his demand if such an allocation is feasible, and with the players each receiving zero otherwise. The strategy sets for players 1 and 2 in the Nash demand game are  $\mathbb{R}_+$ , with payoff function

$$\pi(s_1, s_2) = \begin{cases} (s_1, s_2) & \text{if } (s_1, s_2) \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Figure 7a illustrates this game with strategy sets restricted to  $\{s_1, s'_1\}$  and  $\{f(s_1), f(s'_1)\}$ , where  $s_1 < s'_1$  and hence  $f(s_1) > f(s'_1)$ . The *Nash bargaining solution* (Nash, 1950) selects the (unique) element

$$(s_1^*, s_2^*) = \arg \max_{(s_1, s_2) \in S} s_1 s_2.$$

To apply stochastic stability ideas to the Nash demand game, we assume players 1 and 2 make demands from the finite sets

$$\begin{aligned} 1: & \quad \{\delta, 2\delta, \dots, \bar{s}_1 - \delta\} \cup \{s_1^*\}, \\ 2: & \quad \{f(\delta), f(2\delta), \dots, f(\bar{s}_1 - \delta)\} \cup \{f(s_1^*)\}, \end{aligned}$$

<sup>12</sup> Hart's model differs from Nöldeke and Samuelson in some additional respects that makes a direct comparison difficult. Kuzmics (2002) completes the picture by showing that Hart's result holds in Nöldeke and Samuelson's model when limits are taken as in (9).

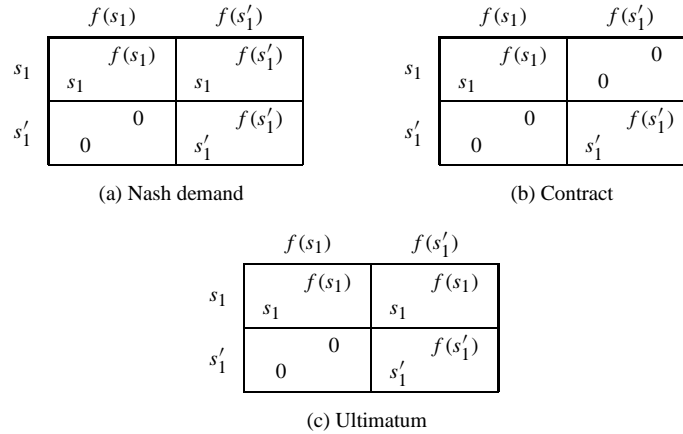


Fig. 7. Pairwise comparisons in bargaining games ( $s_1 < s'_1$ ).

where  $\bar{s}_1$  is the largest feasible utility for player 1 and  $s_1^*$  is player 1's utility in the Nash bargaining solution. Removing the Nash bargaining solution from the set of demands leads only to more cumbersome statements and approximate versions our results.

**Proposition 5.** *The Nash bargaining solution is half-dominant in the Nash demand game with the continuous best-response dynamics. It is therefore the unique stochastically stable equilibrium.*

**Proof.** Consider two pure equilibria,  $(s_1, f(s_1))$  and  $(s'_1, f(s'_1))$ , where  $s_1 < s'_1$ . The pairwise comparison of these two equilibria is illustrated in Fig. 7a. The mixed equilibrium  $(\hat{\sigma}_1, \hat{\sigma}_2)$  of this  $2 \times 2$  game is given by

$$\hat{\sigma}_1 = \frac{f(s'_1)}{f(s_1)}, \quad \hat{\sigma}_2 = 1 - \frac{s_1}{s'_1}.$$

It follows that  $\hat{\sigma}_1 + \hat{\sigma}_2 < 1$  if and only if

$$s'_1 f(s'_1) < s_1 f(s_1).$$

Since the Nash bargaining solution uniquely maximizes the Nash product  $s_1 s_2$  for  $(s_1, s_2)$  in the feasible set, the Nash bargaining solution is half-dominant, and hence, by Proposition 1, uniquely stochastically stable.  $\square$

#### 4.2. Cushioned demand game

Agastya (2001) analyzes a version of the Nash demand game in which the results of a coordination failure are cushioned by splitting some of the unexploited surplus if the players agree on an inefficient outcome. Suppose player 1 demands  $s_1$  and player 2 demands  $f(s'_1)$  for some  $s'_1 > s_1$ , as in the top-right cell of each of the games in Fig. 7. Then we assume the payoffs are a convex combination of  $(s_1, f(s_1))$ , the efficient outcome obtained by holding player 1's demand fixed and inflating 2's demand to the maximum

consistent with feasibility, and  $(s'_1, f(s'_1))$ , the efficient outcome obtained by holding 2's demand fixed and inflating player 1's. Hence, the top-right cell in Fig. 7 yields the payoff

$$\theta(s_1, f(s_1)) + (1 - \theta)(s'_1, f(s'_1))$$

for some  $\theta \in [0, 1]$ . Figure 7c shows the case in which  $\theta$ , the weight attached to  $(s_1, f(s_1))$ , is set equal to one.

Superficially, it looks like player 2 should prefer  $\theta = 1$ , allowing her to appropriate all of any unclaimed surplus. However, setting  $\theta = 1$  puts player 2 in a position equivalent to that of the responder in the Ultimatum Game, in the sense that 2 receives any surplus not claimed by 1 as long as 2 does not scuttle the agreement entirely by demanding too much. Player 2 would accordingly prefer  $\theta = 0$ , which essentially makes her the proposer in an Ultimatum Game.

Anticipating our results, we assume that the finite set of possible demands includes the value  $\tilde{s}_1$  for which

$$\frac{\tilde{s}_1 f'(\tilde{s}_1)}{f(\tilde{s}_1)} = -\frac{\theta}{1 - \theta}. \quad (10)$$

The allocation  $(\tilde{s}_1, f(\tilde{s}_1))$  identifies the point on the boundary of the feasible set at which the ratio of the slope of the tangent to the slope of the ray from the origin equals  $-\theta/(1 - \theta)$ . The case  $\theta = \frac{1}{2}$  takes us back to the Nash bargaining solution. More generally, the solution to (10) is the asymmetric Nash bargaining solution with bargaining powers  $\theta$  and  $1 - \theta$ , i.e., (10) is the first-order condition for maximizing

$$s_1^\theta f(s_1)^{1-\theta}.$$

**Proposition 6.** *The equilibrium  $(\tilde{s}_1, f(\tilde{s}_1))$ , corresponding to the asymmetric Nash bargaining solution with bargaining powers  $(\theta, 1 - \theta)$ , is half-dominant, and hence uniquely stochastically stable, in the cushioned demand game with the continuous best-response dynamics.*

**Proof.** Modifying the proof of the Nash demand game, we now have that

$$\hat{\sigma}_1 = \frac{f(s'_1)}{f(s'_1) + (1 - \theta)(f(s_1) - f(s'_1))}, \quad \hat{\sigma}_2 = 1 - \frac{s_1}{s_1 + \theta(s'_1 - s_1)}.$$

It follows that  $\hat{\sigma}_1 + \hat{\sigma}_2 < 1$  if and only if

$$\frac{s_1}{s'_1 - s_1} \frac{f(s_1) - f(s'_1)}{f(s'_1)} > \frac{\theta}{1 - \theta}.$$

The left side is smallest in the limit as  $s'_1 \rightarrow s_1+$ , allowing us to conclude that the equilibrium satisfying (10) is half-dominant.  $\square$

In the case of the Ultimatum Game with player 2 as responder, or  $\theta = 1$ , the half-dominant equilibrium gives  $f(s_1) = 0$ . We thus get the subgame-perfect equilibrium in which the entire surplus is allocated to player 1.

### 4.3. Contract game

Young (1998) introduces the *contract game*. Suppose that two players must again make demands from a set  $S$  with the properties invoked in the case of the Nash demand game. For a pair of demands to be realized, they must match *exactly*, in the sense that any increase in either demand would yield infeasibility. Every other pair of demands leads to payoffs of zero for both parties. Hence, if  $(s_1, s_2)$  and  $(s'_1, s'_2)$  are both exactly feasible, the contract game restricted to these demands is as in Fig. 7b.

Young interprets this as a game in which two players simultaneously propose contracts. If they propose contracts with identical terms, then the resulting (single) contract is executed. If they propose contracts with different terms, then no deal is struck and each receives a payoff of zero. Notice that the risk-dominant equilibrium of the contract game is the allocation with the higher product of payoffs.

To apply a stochastic stability analysis, we assume that the players' strategy sets are given by

$$\begin{aligned} 1: & \quad \{\delta, 2\delta, \dots, \bar{s}_1 - \delta\} \cup \{s_1^*, \hat{s}_1\}, \\ 2: & \quad \{f(\delta), f(2\delta), \dots, f(\bar{s}_1 - \delta)\} \cup \{f(s_1^*), f(\hat{s}_1)\}, \end{aligned}$$

where  $s_1^*$  is player 1's utility in the Nash bargaining solution and  $\hat{s}_1$  is 1's utility in the Kalai–Smorodinsky solution (Kalai and Smorodinsky, 1975). Removing either of the latter from the set of demands leads to similar but more cumbersome statements of our results. Unlike Young (1998), we simplify the analysis by considering only contracts whose payoffs lie on the Pareto frontier of the bargaining set.

The unique stochastically stable state in Young's learning dynamics is the Kalai–Smorodinsky bargaining solution—a result we confirm holds also for the random best-response dynamics in Section 5.2. However, the same is not true for the continuous best-response dynamics.

**Proposition 7.** *The Nash bargaining solution is half-dominant in the contract game with the continuous best-response dynamics. It is therefore the unique stochastically stable equilibrium.*

**Proof.** The proof for the Nash demand game needs only slight modification. We now have

$$\hat{\sigma}_1 = \frac{f(s'_1)}{f(s_1) + f(s'_1)}, \quad \hat{\sigma}_2 = 1 - \frac{s_1}{s_1 + s'_1}.$$

It follows that  $\hat{\sigma}_1 + \hat{\sigma}_2 < 1$  if and only if

$$s'_1 f(s'_1) < s_1 f(s_1). \quad \square$$

## 5. Random best response in bargaining

### 5.1. Nash demand game

Showing that the Nash bargaining solution is selected under continuous best-response dynamics required a straightforward application of the local resistance test. As noted in Proposition 4, the random best-response dynamics do not typically yield half-dominant equilibria, making the local resistance test less useful. Instead, we exploit the naive minimization test.

**Proposition 8.** *The unique stochastically stable state in the Nash demand game under random best-response dynamics is the Nash bargaining solution.*

**Proof.** To construct the naive minimization tree, we identify, for each population state  $x$  corresponding to a pure equilibrium  $(s_1, f(s_1))$  of the Nash demand game, the equilibrium state that can be reached from  $x$  with the fewest mutations and the proportion of the population that must mutate in order to make such a transition.

Fix a state corresponding to equilibrium  $(s_1, f(s_1))$ . We are interested in the number of mutations required to move to the state corresponding to  $(s'_1, f(s'_1))$ . If  $s'_1 > s_1$ , we must consider the game

	$f(s_1)$	$f(s'_1)$
$s_1$	$f(s_1)$ $s_1$	$f(s'_1)$ $s_1$
$s'_1$	0 0	$f(s'_1)$ $s'_1$

There are then two possibilities. First, mutations to  $s'_1$  in population 1 may make  $f(s'_1)$  a best response in population 2. This requires  $p^+(s_1)$  mutations, where  $(1 - p^+(s_1))f(s_1) = f(s'_1)$ . It is clear that the value of  $p^+(s_1)$  will be minimized if  $f(s'_1)$  is equal to its largest possible value, leading to

$$p^+(s_1) = \frac{f(s_1) - f(s_1 + \delta)}{f(s_1)}. \quad (11)$$

Alternatively, mutations in population 2 may make  $s'_1$  a best response for population 1. This requires  $P^+(s_1)$  mutations, where  $P^+(s_1)s'_1 = s_1$ . Here,  $P^+(s_1)$  is minimized if  $s'_1$  is as large as possible, leading to

$$P^+(s_1) = \frac{s_1}{s'_1 - \delta}.$$

Similarly, we consider a movement to an equilibrium with  $s'_1 < s_1$ , giving the game:

	$f(s_1)$	$f(s'_1)$
$s_1$	$f(s_1)$	0
$s'_1$	$f(s_1)$	$f(s'_1)$

Suppose mutations in population 2 make  $s'_1$  a best response for player 1. This requires  $p^-(s_1)$  mutations, where  $(1 - p^-(s_1))s_1 = s'_1$ , which will be minimized when  $s'_1$  is as large as possible, giving rise to

$$p^-(s_1) = \frac{s_1 - (s_1 - \delta)}{s_1} = \frac{\delta}{s_1}. \tag{12}$$

Alternatively, we could have mutations in population 1 that make  $f(s'_1)$  a best response for population 2. This requires  $P^-(s_1)$  mutations, where  $f(s_1) = P^-(s_1)f(s'_1)$ . We minimize  $P^-(s_1)$  by maximizing  $f(s'_1)$ , giving

$$P^-(s_1) = \frac{f(s_1)}{f(\delta)}.$$

Now compare  $p^+(s_1)$  with  $p^-(s_1 + \delta)$ . We have

$$p^+(s_1) = \frac{f(s_1) - f(s_1 + \delta)}{f(s_1)} > (<) \frac{\delta}{s_1 + \delta} = p^-(s_1 + \delta) \\ \iff s_1 f(s_1) > (<) f(s_1 + \delta)(s_1 + \delta). \tag{13}$$

Hence, the mutation-minimizing movement between  $(s_1, f(s_1))$  and  $(s_1 + \delta, f(s_1 + \delta))$  is the movement in the direction of the allocation with the larger Nash product, i.e., the movement toward the Nash bargaining solution. We next notice that  $p^+(s_1)$  is increasing in  $s_1$  (because  $f$  is convex) while  $p^-(s_1)$  is decreasing in  $s_1$ , allowing us to conclude from (13) that

$$p^+(s_1) = \frac{f(s_1) - f(s_1 + \delta)}{f(s_1)} > (<) \frac{\delta}{s_1} = p^-(s_1) \\ \iff s_1 f(s_1) > (<) f(s_1^*)(s_1^*).$$

Hence, if we allowed only movement between adjacent allocations, the collection of mutation-minimizing transitions would be a directed graph with a single loop containing the Nash bargaining solution, with this loop containing an edge with the maximum resistance over all edges. Call this the *adjacent tree*. However, mutation-minimizing transitions may involve transitions to the extreme allocations represented by the probabilities  $P^+(s_1)$  and  $P^-(s_1)$ .

Suppose first that  $\delta f(\delta) > (\bar{s}_1 - \delta)f(\bar{s}_1 - \delta)$ . Let  $s'_1 \geq s_1^*$  be the smallest value of  $s_1$  for which  $\delta f(\delta) > s'_1 f(s'_1)$ . Then note that

$$P^-(s_1) < p^-(s_1) \iff s_1 f(s_1) < \delta f(\delta),$$

and that

$$\begin{aligned}
P^+(s_1) < p^+(s_1) &\iff s_1 f(s_1) < (\bar{s}_1 - \delta)(f(s_1) - f(s_1 + \delta)) \\
&\implies s_1 f(s_1) < (\bar{s}_1 - \delta)(f(\bar{s}_1 - \delta)).
\end{aligned}$$

Hence, in this case the collection of least-cost transitions involves no probabilities of the form  $P^+(s_1)$ , i.e., no jumps to  $(\bar{s}_1 - \delta, f(\bar{s}_1 - \delta))$ , and modifies the adjacent tree by replacing every transition emanating from each  $(s_1, f(s_1))$  for which  $s_1 \geq s'_1$  with a transition to  $(\delta, f(\delta))$ . The resulting directed graph satisfies the naive minimization principle, featuring a loop that is unique and contains the highest-resistance transition, with the latter having the Nash bargaining solution as its terminus.

Suppose instead that  $\delta f(\delta) < (\bar{s}_1 - \delta)f(\bar{s}_1 - \delta)$ . Then a similar argument shows that the collection of least-cost transitions involves no probabilities of the form  $P^-(s_1)$ , i.e., no jumps to  $(\delta, f(\delta))$ . Similarly, there must exist some  $s'_1 < s_1^*$  such that the collection of least-cost transitions modifies the adjacent tree by replacing every transition emanating from each  $(s_1, f(s_1))$  with  $s_1 \leq s'_1$  with a transition to  $(\bar{s}_1 - \delta, f(\bar{s}_1 - \delta))$ . The resulting directed graph again satisfies the naive minimization principle, featuring a loop that is unique and contains the highest-resistance transition, with the latter having the Nash bargaining solution as its terminus.  $\square$

### 5.2. Cushioned demand game

Recall that the equilibrium  $(\tilde{s}_1, f(\tilde{s}_1))$ , defined by (10), identifies the asymmetric Nash bargaining solution with bargaining powers  $\theta$  for player 1 and  $1 - \theta$  for player 2. A straightforward modification of the proof of Proposition 8 gives:

**Proposition 9.** *The equilibrium  $(\tilde{s}_1, f(\tilde{s}_1))$  is uniquely stochastically stable in the cushioned demand game with the random best-response dynamics.*

### 5.3. Contract game

Young (1998) establishes a more general version of the following:

**Proposition 10.** *For sufficiently small  $\delta$ , the stochastically stable state in the contract game under random best-response dynamics is the Kalai–Smorodinsky solution.*

Young allows a more general class of contracts, including contracts whose payoffs are Pareto dominated by those of other contracts, while also allowing the possibility that there is no contract whose payoffs correspond precisely to the Kalai–Smorodinsky solution. Young shows that, in general, one obtains an approximation of Kalai–Smorodinsky solution.<sup>13</sup>

<sup>13</sup> See Young (1998) for examples and for a formula describing the degree of approximation.

**Proof.** For convenience, denote the largest feasible payoff for player  $i$  by  $\bar{S}_i$ , so that  $\bar{S}_2 = f(\delta)$  and  $\bar{S}_1 = \bar{s}_1 - \delta$ . The Kalai–Smorodinsky solution is that  $(\hat{s}_1, \hat{s}_2)$  satisfying

$$\frac{\hat{s}_1}{\hat{s}_2} = \frac{\bar{S}_1}{\bar{S}_2}.$$

Recall that we have assumed that this is a feasible outcome.

We again begin with an eye toward applying the naive minimization test. We first identify the collection of least-cost transitions. Consider a contract game of the form

	$s_2$	$s'_2$
$s_1$	$s_2$ $s_1$	$0$ $0$
$s'_1$	$0$ $0$	$s'_2$ $s'_1$

where  $s_1 < s'_1$  and  $s_2 > s'_2$ . Then the smallest proportion of mutants required to move from equilibrium  $(s_1, s_2)$  to  $(s'_1, s'_2)$  will be mutants who introduce strategy  $s'_2$  into population 2, requiring proportion

$$\frac{s_1}{s_1 + s'_1}$$

of population 2 to be such mutants. The smallest proportion of mutants required to move from equilibrium  $(s'_1, s'_2)$  to  $(s_1, s_2)$  will be mutants who introduce strategy  $s_1$  into population 1, requiring proportion

$$\frac{s_2}{s_2 + s'_2}$$

of population 1 to be such mutants. The principle here is that fewest mutations are required in that population whose payoff stands to be increased by the transition. In addition, it is apparent from these calculations that fewer mutations will be required for a transition the larger is the terminal payoff received by the population into which mutants are being introduced. This in turn implies that the least-cost transitions will always be to one of the extreme states  $(\bar{S}_1, f(\bar{S}_1))$  or  $(\delta, \bar{S}_2)$ .

For each state (other than the extreme states themselves), we thus have two possible cheapest transitions, one to each of the extreme states. Which will be cheapest? From an arbitrary state  $(s_1, s_2)$  the mutation costs of transitions to the extreme states are

$$\frac{s_1}{\bar{S}_1 + s_1} \quad \text{and} \quad \frac{s_2}{\bar{S}_2 + s_2}.$$

It is then straightforward to calculate that the collection of cheapest transitions connect all those allocations in which player  $i$  gets less than  $i$ 's share in the Kalai–Smorodinsky solution to the extreme point in which  $i$  gets the relatively larger payoff. This least-cost collection of transitions includes a unique cycle, connecting the two extreme states to each other.

Unfortunately, we cannot apply the naive minimization principle directly, since this cycle involves the least rather than most expensive transitions in the tree. The most

expensive transition begins with the Kalai–Smorodinsky solution, from which it is equally costly to move to either extreme state. However, the least-cost collection of transitions tells us a great deal about the structure of the minimal tree. There will be a path connecting one of the extreme states to the root of the tree, where this path does not contain the other extreme state. Every state that is not contained in this path will be the origin of a transition to one of the extreme states, with one of the extreme states connected to the root through the other extreme state.

Identifying the minimal tree thus requires finding a path, beginning with an extreme state and ending with the root of the tree, that maximizes the difference between the mutation costs of the transitions actually used in constructing the path and the mutation costs of the least-cost transitions emanating from the states on this path. One possibility is to choose an extreme state as the root. However, this eliminates a relatively low-cost transition, from one extreme state to another. A tedious but straightforward calculation, presented in Appendix A, shows that this problem is solved by a path that removes the transition emanating from the Kalai–Smorodinsky solution, and alters the transition from one of the extreme states to have the Kalai–Smorodinsky solution as its destination. This allows the elimination of the most expensive least-cost transition, that beginning from the Kalai–Smorodinsky solution.  $\square$

We can illustrate this result. Suppose there are three efficient allocations, given by (5, 10), (6, 8) and (13, 1). The contract game is then:

	10	8	1
5	5 10	0 0	0 0
6	0 0	8 6	0 0
13	0 0	0 0	1 13

The following identifies the (approximate) proportion of one of the populations that must mutate in order to make a transition from the equilibrium shown in each row to the equilibrium in each column:

	(5, 10)	(6, 8)	(13, 1)
(5, 10)		0.45	0.28
(6, 8)	0.44		0.32
(13, 1)	0.09	0.11	

The Nash bargaining solution is (5, 10). A chain of pairwise risk-dominance arguments leads to this solution, in the sense that (5, 10) risk dominates (6, 8) and (6, 8) risk dominates (13, 1). However, the outcome closest to the Kalai–Smorodinsky solution is (6, 8). Attaching to each state its least cost transition gives transitions from the extreme

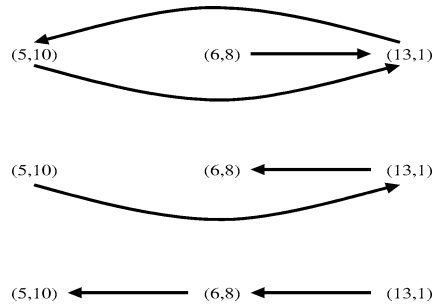


Fig. 8. Directed graph produced by naive minimization test (top); minimal tree, with Kalai–Smorodinsky solution as root (middle); least-cost tree when only local mutations are allowed (cf. Section 7), with Nash bargaining solution as its root (bottom).

states  $(5, 10)$  and  $(13, 1)$  to each other and a transition from  $(6, 8)$  to  $(13, 1)$ . The least-cost tree connects  $(5, 10)$  to  $(13, 1)$  to  $(6, 8)$ . Figure 8 illustrates.

The key characteristic of the contract game is thus that cheapest transitions are to the extreme points in the state space. It is then no surprise that these extreme points play a role in determining the stochastically stable state and that the stochastically stable state turns out to be the Kalai–Smorodinsky solution.

#### 5.4. Comparison

Continuous and random best-response dynamics give equivalent outcomes in the Nash demand game, though we arrive at these results via different arguments. In contrast, we find different outcomes for the contract game. What accounts for this difference? In both cases, the pruning principle and the bandwagon property allow us to reduce the quest for a mutation-minimizing tree to an examination of the mutation costs of transitions in a series of  $2 \times 2$  games. But which  $2 \times 2$  games are relevant? The continuous best-response dynamics allows us to circumvent this problem by finding a half-dominant equilibrium, the Nash bargaining solution, in both the Nash demand game and contract game.

The random best-response dynamics do not yield a half-dominant equilibrium, requiring that we look at the  $2 \times 2$  games more carefully. However, the off-diagonal payoffs in the Nash demand game ensure both that risk dominance provides a guide to mutation minimization in transitions between *adjacent* allocations, and also that adjacent transitions are the relevant transitions for all but a few relatively extreme states. We then again arrive at the Nash bargaining solution. In contrast, the zero off-diagonal elements in the contract game ensure that adjacent transitions are virtually never relevant. Instead, cheapest transitions are always to extreme states. Finding the least-cost tree reduces to finding the state  $x$  that maximizes the difference between the cost of moving from  $x$  to an extreme state and the cost of moving from the extreme case to  $x$ . This turns out to be the Kalai–Smorodinsky solution.

## 6. Links with other bargaining approaches

### 6.1. Risk dominance

An equilibrium  $(s_1, s_2)$  in a  $2 \times 2$  game with strategy sets  $\{s_1, s'_1\}$  and  $\{s_2, s'_2\}$  is *risk-dominant* (Harsanyi and Selten, 1988) if

$$\hat{\sigma}_1 + \hat{\sigma}_2 \leq 1,$$

where  $\hat{\sigma}_i$  is the probability of strategy  $s_i$  that makes player  $j$  indifferent between  $s_j$  and  $s'_j$ , if such a probability exists, and is otherwise zero.

We can use this notion to gain some insight into why the Nash bargaining solution is stochastically stable under the continuous and random best-response dynamics. First, Appendix A proves:

**Proposition 11.** (11.1) *The risk-dominant equilibrium in the Nash demand game in which player 1 and player 2 are restricted to strategies  $\{s_1, s'_1\}$  and  $\{f(s_1), f(s'_1)\}$  is the allocation with the larger Nash product.*

(11.2) *An allocation  $(s_1^*, f(s_1^*)) \in S^*$  is the Nash bargaining solution if and only if it is the risk-dominant equilibrium in every Nash demand game in which player 1 and player 2 are restricted to strategies  $\{s_1^*, s'_1\}$  and  $\{f(s_1^*), f(s'_1)\}$*

Hence, the Nash bargaining solution  $(s_1^*, f(s_1^*))$  is risk dominant in every  $2 \times 2$  game that can be constructed by confining agents to demands corresponding either to the Nash bargaining solution or to any single alternative feasible allocation. Proposition 5 is then proved by noting that, under the continuous best-response dynamics, this implies that the Nash bargaining solution is half dominant, leading immediately to stochastic stability.

Half dominance fails with random best responses, but we can gain some insight into why the naive minimization procedure works:

**Corollary 2.** *The risk-dominant equilibrium in the Nash demand game with strategy sets  $\{s_1, s_1 + \delta\}$ ,  $\{f(s_1), f(s_1 + \delta)\}$  is  $(s_1 + \delta, f(s_1 + \delta))$  if  $s_1 + \delta \leq s_1^*$  and is  $(s_1, f(s_1))$  if  $s_1 \geq s_1^*$ .*

Proposition 8 uses this result to show that, in the case of the random best-response dynamics, the naive minimization test yields a tree with the Nash bargaining solution as its root. The two key transition probabilities away from an allocation  $(s_1, f(s_1))$ , under best-response dynamics, are  $p^+(s_1)$  and  $p^-(s_1)$  (cf. (11), (12)). These probabilities share the feature that each measures the number of mutations required to induce an opponent to accept a slightly smaller payoff, with certainty, rather than an uncertain prospect of a slightly larger payoff. The key property of the Nash bargaining solution is that it is “easier,” measured in terms of the number of mutations, to move in the direction of risk dominance,

and hence easier to induce player  $i$  (rather than  $j$ ) to make this concession if  $i$ 's allocation exceeds  $i$ 's share in the Nash bargaining solution.<sup>14</sup>

### 6.2. Rubinstein, Safra, and Thomson

Rubinstein et al. (1992) show that an allocation  $(s_1^*, s_2^*)$  is the Nash bargaining solution if and only if, for any alternative allocation  $(s_1, s_2) \in S$ , probability  $p \in [0, 1]$ , and players  $i, j \in \{1, 2\}$  with  $i \neq j$ ,

$$ps_i > s_i^* \quad \Rightarrow \quad ps_j^* \geq s_j. \quad (14)$$

The interpretation is that if player  $i$  is willing to risk disagreement with probability  $1 - p$  to induce alternative  $(s_1, s_2)$  rather than  $(s_1^*, s_2^*)$ , then player  $j$  must be willing to risk disagreement with probability  $1 - p$  in order to retain  $(s_1^*, s_2^*)$  rather than switch to  $(s_1, s_2)$ . We can use the risk-dominance characterization of the Nash bargaining solution to show:

**Proposition 12.** *An allocation  $(s_1^*, s_2^*)$  satisfies the Rubinstein, Safra and Thomson criterion (14) for the Nash bargaining solution with respect to alternative  $(s_1, s_2)$  if and only if  $(s_1^*, s_2^*)$  is risk dominant in the Nash demand game with strategy sets  $\{s_1^*, s_1\}$  and  $\{s_2^*, s_2\}$ .*

### 6.3. Zeuthen

We can similarly interpret Harsanyi's (1977, pp. 149–166) interpretation of a solution to the bargaining problem whose motivation he attributes to Zeuthen. A simplified version of Harsanyi's bargaining process is as follows. Let players 1 and 2 be initially characterized by allocations  $s^1(0)$  and  $s^2(0)$ , where  $s^i(0)$  is the extreme allocation in which player  $i$  receives the largest payoff. We will think of the allocations associated with the two agents as their current proposed settlements.

In period  $t$  the agents are characterized by allocations  $s^1(t)$  and  $s^2(t)$ . In each period, one of the agents makes a concession, a process which moves their associated allocation along the feasible frontier in the direction of the other agent's allocation. The process halts when the two allocations coincide. The concession in period  $t$  is made by the player "less willing to face a conflict." To make this phrase meaningful, let  $p_i$  be the probability that would make player  $i$  indifferent between  $j$ 's proposed settlement and a lottery that gives disagreement with probability  $p_i(t)$  and  $i$ 's proposed settlement with probability  $1 - p_i(t)$ , i.e.,

$$(1 - p_i)s_i^i(t) = s_i^j(t).$$

The player with the lower value of  $p_i(t)$  is deemed the player least willing to risk disagreement and hence the player who must make a concession in period  $t$ . It is then

<sup>14</sup> Similar considerations lie behind Nash's (1953) demonstration that the set of Nash equilibria of a perturbed version of the Nash demand game converges to the Nash bargaining solution as the perturbations approach zero.

straightforward that this process must converge to the Nash bargaining solution, since we have

$$p_1(t) < p_2(t) \implies \frac{s_1^1(t) - s_1^2(t)}{s_1^1(t)} < \frac{s_2^2(t) - s_2^1(t)}{s_2^2(t)},$$

or

$$s_1^1(t)s_2^1(t) < s_1^2(t)s_2^2(t),$$

which is to say that the player whose current allocation has the smallest Nash product makes the next concession. We have thus established:

**Proposition 13.** *Player  $i$  makes the period- $t$  concession in the Zeuthen process if and only if  $i$ 's current allocation yields the smaller Nash product, causing the process to converge to the Nash bargaining solution.*

#### 6.4. Disagreement point

Nothing in our analysis hinges upon the players' payoffs being zero in the event that their demands are infeasible. If instead we assumed that infeasible demands give rise to payoffs  $(\underline{s}_1, \underline{s}_2)$ , then the results given above continue to hold with utilities  $s_1$  and  $s_2$  replaced by  $s_1 - \underline{s}_1$  and  $s_2 - \underline{s}_2$ .

The take-it-or-leave-it nature of the Nash demand game typically makes the nature of the disagreement option  $(\underline{s}_1, \underline{s}_2)$  unambiguous. In more complicated games, such as Rubinstein's alternating-offers game, specifying  $(\underline{s}_1, \underline{s}_2)$  requires more care (Binmore et al., 1986).

#### 6.5. Single population models

When all the players are drawn from a single large population and assigned at random to a player role in a *symmetric*  $2 \times 2$  game, the details of the underlying dynamics are irrelevant, provided it is monotone (Samuelson and Zhang, 1992). As a result, the continuous best-response and random best-response dynamics give equivalent results in such games. This explains why the results of Kandori et al.'s (1993) are not very sensitive to the specification of the dynamics.

Our bargaining models are asymmetric. There are two possibilities for a single-population analysis of such a game. First, the players have identical strategy sets, allowing us to assume that agents use the same strategy, no matter which player role they occupy. Alternatively, we can assume that agents condition their strategy on their player role. In the former case, the continuous best-response and random best-response dynamics give equivalent results, leading to the egalitarian bargaining solution (Young, 1993a). In the latter case, we essentially return to the two population formulation.

## 7. Discussion

One purpose of evolutionary models is to identify and make precise our intuitions about the factors that make some equilibria more interesting than others. Our evolutionary analysis of equilibrium selection draws attention to two factors. First, equilibrium selection can depend upon whether mutations or random variations in the rates at which people learn are the more likely source of perturbations in the strategy adjustment process that potentially switches play between equilibria. The case in which variations in learning are relatively important, presumably because the relevant populations are relatively small, is captured by the random best-response dynamics. In the resulting model, with mutation rates going to zero for a fixed population size, any asymmetric combination of strategy adjustments, no matter how unlikely, becomes much more likely than mutations. The result is that jumps to extremely asymmetric allocations, accomplished by a small burst of mutations in one population followed by a long sequence of learning adjustments concentrated in the other population, become relatively likely in the contract game. The resulting importance of the extremal allocations leads to the selection of the Kalai–Smorodinsky solution.

The case in which mutations are of paramount importance, presumably because idiosyncratic learning paths are quite unlikely in large populations, is captured by the continuous best-response dynamics. We capture this by examining first the limit as the population grows large, and then as the mutation rate shrinks to zero. Here, we arrive at the Nash bargaining solution in the contract game.

Second, equilibrium selection can depend upon which mutations are most likely. If we allow only local mutations, in the sense that mutations can only move players between adjacent allocations, then the outcome will be the Nash bargaining solution, regardless of whether the pairings are done via the Nash demand game or the contract game and regardless of the learning dynamics that we have considered. If nonlocal mutations are allowed, then we continue to select the Nash bargaining solution when in the Nash demand game. However, we get the Kalai–Smorodinsky solution from random best-response dynamics in the contract game, with jumps to and from extreme allocations playing a key role in this result.

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## Appendix A.

### A.1. Proof of Proposition 10

The cost-minimizing tree must feature a transition from one (and only one, since otherwise we have a loop) extreme state to the other, since otherwise both extreme states must lie on paths to the root, and the total cost could be reduced by eliminating the transition emanating from one extreme state and replacing it with a transition to the other extreme state. It sacrifices no generality to assume that there is a transition from  $(\bar{s}_1, f(\bar{s}_1))$  to  $(\delta, \bar{s}_2)$ .

We now show that the transition from  $(\delta, \bar{S}_2)$  in the cost-minimizing tree must go directly to the root of the tree. Suppose it does not, and that there is a transition from  $(\delta, \bar{S}_2)$  to  $(s_1, f(s_1))$  with  $s_1 < s'_1$ , where  $(s'_1, s'_2)$  is the root. Then costs could be reduced by replacing this transition with the transition from  $(\delta, \bar{S}_2)$  to  $(s'_1, s'_2)$ , a contradiction.

Next, suppose there is a transition  $(\delta, \bar{S}_2) \rightarrow (s_1, f(s_1))$  with  $s_1 > s'_1$ . Then there must be a further transition from  $(s_1, f(s_1))$ , say to  $(s'_1, f(s'_1))$ . If  $s_1 < s'_1$ , then costs could be reduced by replacing  $(\delta, \bar{S}_2) \rightarrow (s_1, f(s_1))$  with  $(\delta, \bar{S}_2) \rightarrow (s'_1, f(s'_1))$ , a contradiction. Assume then that  $s_1 > s'_1$ . Then there are transitions  $(\delta, \bar{S}_2) \rightarrow (s_1, f(s_1))$ ,  $(s_1, f(s_1)) \rightarrow (s'_1, f(s'_1))$  with  $s'_1 < s_1$ , and (by assumption)  $(\bar{S}_1, f(\bar{S}_1)) \rightarrow (\delta, \bar{S}_2)$ . The cost of these three transitions is given by

$$\frac{\delta}{\delta + s_1} + \frac{f(s_1)}{f(s_1) + f(s'_1)} + \frac{f(\bar{S}_1)}{f(\bar{S}_1) + \bar{S}_2}. \quad (15)$$

Deleting these three transitions and adding  $(\delta, \bar{S}_2) \rightarrow (\bar{S}_1, f(\bar{S}_1))$ ,  $(\bar{S}_1, f(\bar{S}_1)) \rightarrow (s'_1, f(s'_1))$ , and  $(s_1, f(s_1)) \rightarrow (\delta, \bar{S}_2)$  yields an alternative tree. We argue that this new tree has lower cost. The three transitions added have a total cost of

$$\frac{\delta}{\delta + \bar{S}_1} + \frac{f(\bar{S}_1)}{f(\bar{S}_1) + f(s'_1)} + \frac{f(s_1)}{f(s_1) + \bar{S}_2}. \quad (16)$$

We need to show that the sum in (15) exceeds that in (16). Since

$$\frac{\delta}{\delta + s_1} > \frac{\delta}{\delta + \bar{S}_1},$$

it suffices to show that

$$\frac{f(s_1)}{f(s_1) + f(s'_1)} + \frac{f(\bar{S}_1)}{f(\bar{S}_1) + \bar{S}_2} > \frac{f(\bar{S}_1)}{f(\bar{S}_1) + f(s'_1)} + \frac{f(s_1)}{f(s_1) + \bar{S}_2}. \quad (17)$$

This follows from the inequalities  $\bar{S}_2 > f(s'_1) > f(s_1) > f(\bar{S}_1)$ , as may be verified by direct calculation.

Our conclusion is that there must be a transition from  $(\delta, \bar{S}_2)$  to the root of the tree. All other transitions must then be least-cost transitions. It remains to identify the root of the tree. We break this into two cases.

First, suppose there is a transition  $(\delta, \bar{S}_2) \rightarrow (s_1, f(s_1))$  for some  $s_1$  no larger than the Kalai–Smorodinsky share for player 1. Compared to the naive minimization transitions, this candidate least-cost tree has deleted the transitions  $(\delta, \bar{S}_2) \rightarrow (\bar{S}_1, f(\bar{S}_1))$  and  $(s_1, f(s_1)) \rightarrow (\bar{S}_1, f(\bar{S}_1))$ . Compared to the naive minimizations, we then have a cost change of

$$\frac{\delta}{\delta + s_1} - \frac{s_1}{s_1 + \bar{S}_1} - \frac{\delta}{\delta + \bar{S}_1}. \quad (18)$$

Given that  $s_1$  is no larger than player 1's share in the Kalai–Smorodinsky solution, (18) is minimized by setting  $(s_1, f(s_1))$  equal to the Kalai–Smorodinsky solution.

Second, suppose there is a transition to a root  $(s'_1, f(s'_1))$  that gives player 1 a larger share than the Kalai–Smorodinsky solution. Compared to the naive minimization transitions, we have added  $(\delta, \bar{S}_2) \rightarrow (s'_1, f(s'_1))$  and deleted  $(s'_1, f(s'_1)) \rightarrow (\delta, \bar{S}_2)$  and  $(\delta, \bar{S}_2) \rightarrow (\bar{S}_1, f(\bar{S}_1))$ , where  $s'_1$  is strictly larger than 1's share in the Kalai–Smorodinsky solution. The change in cost is

$$\frac{\delta}{\delta + s'_1} - \frac{\delta}{\delta + \bar{S}_1} - \frac{f(s'_1)}{f(s'_1) + \bar{S}_2}.$$

We will show that this is larger than (18), where  $(s_1, f(s_1))$  in (18) is taken to be the Kalai–Smorodinsky solution. The relevant comparison is

$$\frac{\delta}{\delta + s'_1} - \frac{\delta}{\delta + \bar{S}_1} - \frac{f(s'_1)}{f(s'_1) + \bar{S}_2} > \frac{\delta}{\delta + s_1} - \frac{s_1}{s_1 + \bar{S}_1} - \frac{\delta}{\delta + \bar{S}_1}.$$

Eliminating the common term, we have

$$\frac{\delta}{\delta + s'_1} - \frac{f(s'_1)}{f(s'_1) + \bar{S}_2} > \frac{\delta}{\delta + s_1} - \frac{s_1}{s_1 + \bar{S}_1}.$$

Because  $(s_1, f(s_1))$  is the Kalai–Smorodinsky solution, we have

$$\frac{s_1}{s_1 + \bar{s}_1} = \frac{f(s_1)}{f(s_1) + \bar{s}_2}.$$

After substitution and rearrangement, we obtain

$$\frac{\delta}{\delta + s'_1} + \frac{f(s_1)}{f(s_1) + \bar{s}_2} > \frac{\delta}{\delta + s_1} + \frac{f(s'_1)}{f(s'_1) + \bar{s}_2}.$$

Rearranging and simplifying, it can be shown that this is equivalent to

$$\bar{s}_2(\delta + s'_1)(\delta + s_1)(f(s_1) - f(s'_1)) > \delta(f(s_1) + \bar{s}_2)(\bar{s}_2 + f(s'_1))(s'_1 - s_1), \tag{19}$$

which clearly holds for all sufficiently small  $\delta$ .

A.2. Proof of Proposition 11

Consider the restricted Nash demand game given by

	$s_2$	$s'_2$
$s_1$	$s_2$ $s_1$	$s'_2$ $s_1$
$s'_1$	0 0	$s'_2$ $s'_1$

It is immediate that  $(s_1, s_2)$  is a candidate for the Nash bargaining solution if and only if it is an equilibrium of this game.

Harsanyi and Selten’s (1988, p. 87) notion of risk dominance for general  $2 \times 2$  games is that  $(s_1, s_2)$  maximize the product of deviation losses, which in this case is

$$s_1(s_2 - s'_2) > (s'_1 - s_1)(s'_2) \quad \text{or} \quad s_1 s_2 > s'_1 s'_2.$$

A.3. Proof of Proposition 12

**Case 1.** Suppose that  $(s_1^*, s_2)$  and  $(s_1, s_2^*)$  are both feasible. Consider the game:

	$s_2^*$	$s_2$
$s_1^*$	$s_2^*$ $s_1^*$	$s_2$ $s_1^*$
$s_1$	$s_2^*$ $s_1$	$s_2$ $s_1$

In this case,  $(s_1^*, s_2^*)$  is risk dominant if and only if

$$s_1^* \geq s_1, \tag{20}$$

$$s_2^* \geq s_2, \tag{21}$$

where at least one inequality must be strict (since by hypothesis  $(s_1^*, s_2^*) \neq (s_1, s_2)$ ). If (20)–(21) both fail, then (14) fails for  $p$  sufficiently close to one. If (20) and (21) both hold, then (14) holds for all  $p$ . Finally, (20) can fail and (21) hold (the reverse case is analogous) if and only if (14) fails for the alternative of  $(s_1, s_2^*)$ . Hence, risk dominance in the restricted game is equivalent to satisfying the Rubinstein, Safra, and Thomson criterion in this case.

**Case 2.** Suppose that neither  $(s_1^*, s_2)$  nor  $(s_1, s_2^*)$  is feasible. Then we have:

	$s_2^*$	$s_2$
$s_1^*$	$s_1^*$ $s_2^*$	0
$s_1$	0	$s_2$ $s_1$

Let

$$p_1 s_2^* = (1 - p_1) s_2, \quad p_2 s_1^* = (1 - p_2) s_1,$$

or

$$\frac{p_1}{1 - p_1} s_2^* = s_2, \quad s_1^* = \frac{1 - p_2}{p_2} s_1.$$

Then  $(s_1^*, s_2^*)$  is risk dominant if  $p_1 + p_2 \leq 1$ . Now note that a violation of (14) with  $i = 1$  (the case of  $i = 2$  is analogous) exists if and only if there exists a certain  $p$  with

$$\frac{1 - p_2}{p_2} < p < \frac{p_1}{1 - p_1}.$$

Such a violation will be impossible if and only if

$$\frac{p_1}{1 - p_1} \leq \frac{1 - p_2}{p_2} \quad \text{or, simplifying,} \quad p_1 + p_2 \leq 1,$$

which is to say that  $(s_1^*, s_2^*)$  is risk dominant. Again, the Rubinstein, Safra, Thomson criterion is equivalent to  $(s_1^*, s_2^*)$  being risk-dominant.

**Case 3.** Suppose that  $(s_1^*, s_2)$  is feasible and  $(s_1, s_2^*)$  is not. (The reverse assumption yields an analogous case.) Then we have:

	$s_2^*$	$s_2$
$s_1^*$	$s_1^*$ $s_2^*$	$s_2$
$s_1$	0	$s_2$ $s_1$

Let

$$p_1 s_2^* = s_2, \quad s_1^* = (1 - p_2) s_1,$$

or

$$s_2^* = \frac{1}{p_1} s_2, \quad \frac{1}{1 - p_2} s_1^* = s_1.$$

Then  $(s_1^*, s_2^*)$  is risk-dominant if  $p_1 + p_2 \leq 1$ . A violation of (14) with  $i = 1$  exists if and only if there exists a certain  $p$  with

$$1 - p_2 < p < p_1.$$

Such a violation will be impossible if and only if

$$p_1 \leq 1 - p_2 \quad \text{or} \quad p_1 + p_2 \leq 1,$$

which is to say that  $(s_1^*, s_2^*)$  is risk dominant. Similarly, a violation of (14) with  $i = 2$  exists if and only if there exists a certain  $p$  with

$$\frac{1}{p_1} < p < \frac{1}{1 - p_2}.$$

Such a violation will be impossible if and only if

$$\frac{1}{1 - p_2} \leq \frac{1}{p_1} \quad \text{or} \quad p_1 + p_2 \leq 1,$$

which is to say that  $(s_1^*, s_2^*)$  is risk dominant.  $\square$

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