Forward Looking Estimates of Interest Rate Distributions *

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Abstract

This paper reviews methods for extracting both risk-neutral and physical density forecasts for interest rates. It presents some applications, with particular focus on issues pertaining to forward guidance and the zero lower bound. Several important applied questions in macroeconomics and monetary economics can be very directly addressed using the wealth of information in interest rate derivative securities.

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1 Introduction

Central banks, academics and the press routinely use the yield curve to measure expectations of the future path of interest rates, although they are increasingly aware that the task is complicated by the presence of time-varying term premia (Shiller, Campbell, and Schoenholtz, 1983; Kim and Wright, 2005; Cochrane and Piazzesi, 2005; Adrian, Crump, and Moench, 2013). But much more can be done. Financial asset prices help us to measure uncertainty about future interest rates and higher moments—in fact, they can give us entire probability density functions, although these are again complicated by risk premia. The technology for forward-looking estimates of interest rate distributions is well established in the finance literature, but is underutilized by monetary and macroeconomists. It can, in particular, be used to answer questions about the efficacy of forward guidance and the probability of being at the zero lower bound, or of negative interest rates, at dates in the future. It can be used to assess how the zero lower bound affects longer-term interest rates.

This paper reviews methods for constructing forward looking estimates of interest rate distributions. I will consider the distributions implied by the prices of options and other derivatives. These give the state price densities, i.e. distributions under the risk neutral, or more precisely, under the forward measure. Their interpretation is complicated by the presence of risk premia. I will also consider methods to construct physical density forecasts. Given the availability of actively traded interest rate derivatives, risk-neutral pdfs for future nominal interest rates can be constructed day-by-day with minimal modeling assumptions and negligible measurement error. Alas, it is not so easy for the physical density forecasts. These are normally constructed based on historical time series patterns of interest rates, and are plagued by model and parameter uncertainty and by structural instability. In limited cases, surveys provide an alternative source of information on physical density forecasts, that do not suffer from these problems.

The plan for the remainder of this paper is as follows. In section 2, I review methods of constructing risk-neutral pdfs. This is a brief review of an enormous literature—far more comprehensive treatments from a finance practitioner's perspective are available in Rebonato (2003) and Brigo and Mercurio (2006). In section 3, I turn to physical density forecasts, and discuss the pricing kernel that links the risk-neutral and physical densities. Section 4 applies the methods to Eurodollar options and other interest rate derivatives over recent years, with particular emphasis on questions pertaining to the zero lower bound (ZLB). This section is intended to highlight the potential uses of the techniques discussed in this paper in macroeconomics and central banking. Section 5 concludes.

2 Risk-neutral pdfs

Define $P_t(n)$ as the price at time t of a n-period zero coupon bond that pays \$1 at time $t+n$ and let $y_t(n) = -\frac{1}{n}\log(P_t(n))$ denote its yield. Let

$$f_t(n_1, n_2) = \frac{(n_2 - t)y_t(n_2 - t) - (n_1 - t)y_t(n_1 - t)}{n_2 - n_1} \tag{2.1}$$

denote a forward interest rate from time n_1 to time n_2 ($t \leq n_1 < n_2$).

Suppose that we have options with payoffs that are tied to $y_\tau(n)$ an n-period interest rate at time $\tau \geq t$. The payoffs are

$$max(y_\tau(n) - K, 0) \tag{2.2}$$

for different choices of the strike price K. For example, if n is 3 months, this corresponds to a Eurodollar option that is based on a three-month LIBOR interest rate[1].

The valuation of this asset at time t is:

$$E_t(M(t,\tau)\max(y_\tau(n) - K, 0)) \tag{2.3}$$

where E_t denotes the physical expectation and $M(t_1, t_2)$ is the pricing kernel from time t_1 to time t_2. The valuation can alternatively be written as:

$$P_t(\tau - t)E_t^*(\max(y_\tau(n) - K, 0)) \tag{2.4}$$

[1] Strictly, Eurodollar options settle not to the interest rate but to 100 minus the interest rate. A call/put Eurodollar option is a bet on falling/rising interest rates, respectively. But we can think of them as derivatives with payoffs of the form of equation (2.2).

where E_t^* denotes the expectation under the forward risk-neutral measure[2]. Henceforth, I refer to the forward risk-neutral measure as the risk-neutral measure, or Q-measure. The objective is to reverse-engineer the Q-measure probability density for the future interest rate $y_\tau(n)$ from the observed price of the options.

2.1 Other derivatives

Many more interest rate derivatives exist, but these can all be priced using the same principles. Here is a list of some of the most important, that are discussed in much depth in references including Brigo and Mercurio (2006) and Veronesi (2010).

1. Interest rate caps are contracts that at various future dates pay the amount that the interest rate exceeds a strike price on a notional underlying principal if the interest rate exceeds that strike price, and zero otherwise. These are effectively bundles of call options on the interest rate, with payoffs of the form in equation (2.2).

2. Options on Treasury futures confer the right, but not the obligation, to enter into a Treasury futures contract. They are traded on the CME and are American options (early exercise is possible). Treasury futures contracts are complicated, but are structured in such a way that a long position in a Treasury futures contract can be roughly thought of as a contract to buy the shortest maturity security in the delivery basket[3], and so the futures option is approximately an option on that security. The MOVE index is an index of implied volatilities on two-, five-, ten- and thirty- year Treasury futures contracts, constructed by Merrill Lynch, and is the equivalent of the VIX for interest rate uncertainty. Since 2011, the CME has traded options on Treasury futures that expire each Friday afternoon[4]. Though new, they are quite liquid, and can be used to

[2] The valuation of the security under the ordinary risk-neutral measure would be the expectation of $exp(-\int_t^{t+\tau} r(s)ds)\max(y_\tau(n)-K,0)$ where $r(s)$ is the instantaneous interest rate. The discount factor $exp(-\int_t^{t+\tau} r(s)ds)$ is random and cannot be pulled through the expectations operator. The forward risk-neutral measure (Jamshidian, 2007; Jarrow, Li, and Zhao, 2007; Li and Zhao, 2009) instead takes the $(\tau-t)$-period zero coupon bond as the numeraire, which is known at time t and so can be pulled outside the expectations operator.

[3] This statement is true for US Treasury futures so long as the level of interest rates is below 6 percent, which has been true for a very long time.

[4] The holder must enter into a futures position by the expiration day, or else the option expires unused.

measure uncertainty about specific risk events, like employment reports (Kim and Wright, 2014).

3. A *payer* swaption gives the holder the right, but not the obligation, to enter into a swap contract where they pay a particular fixed rate. A *receiver* swaption gives the holder the right, but not the obligation, to enter a swap contract where they receive a particular fixed rate. Swaptions are typically European[5]. If the swaption expires at time τ and the underlying swap contract calls for R payments every Δ years on a notional underlying principal of L, if the swaption has a strike price of i_S and if the swap rate at expiration is i_F, then the value of a receiver swaption at expiration is $L\Delta \max(i_F - i_S, 0)\Sigma_{i=1}^{R} P_\tau(i\Delta)$ which is a rescaled version of the payoff in equation (2.2).

2.2 Black Model

For an option with the payoffs in equation(2.2), a simple approach to reverse-engineering the risk-neutral pdf is to posit that the forward rate $f_t(\tau, \tau+n)$ follows the following process under the Q-measure:

$$\frac{df_t(\tau, \tau+n)}{f_t(\tau, \tau+n)} = \sigma dW_t \tag{2.5}$$

where W_t is a standard Brownian motion. Noting that $f_\tau(\tau, \tau+n) = y_\tau(n)$, it follows using Ito's lemma that:

$$\log y_\tau(n) \sim N(\log(f_t(\tau, \tau+n)) - \frac{1}{2}\sigma^2\tau^*, \sigma^2\tau^*) \tag{2.6}$$

where $\tau^* = \tau - t$ is the time to maturity. Under this premise, Black (1976) gives an analytical expression for equation (2.4) at time t. Given the price of a single option, this can be inverted to obtain the implied volatility, σ. A 95% confidence interval for $y_\tau(n)$ can then be constructed as:

$$[f_t(\tau, \tau+n)\exp(-\frac{1}{2}\sigma^2\tau^* - 1.96\sigma\sqrt{\tau^*}), f_t(\tau, \tau+n)\exp(-\frac{1}{2}\sigma^2\tau^* + 1.96\sigma\sqrt{\tau^*})] \tag{2.7}$$

The implied volatility in equation (2.5) is not directly a measure of uncertainty about

[5] Bermudan swaptions are also quite common. A Bermudan swaption can be exercised on certain specified dates only.

the level of interest rates. Uncertainty about the level of interest rates can be measured by the width of a 95% confidence interval for $y_\tau(n)$, which is:

$$f_t(\tau, \tau+n) \exp(-\frac{1}{2}\sigma^2\tau^*)[\exp(1.96\sigma\sqrt{\tau^*}) - \exp(-1.96\sigma\sqrt{\tau^*})] \tag{2.8}$$

The assumption in equation (2.5) implies that interest rates can never go negative. Until recently, this seemed quite uncontroversial. But many benchmark interest rates are negative today, and that cannot be accommodated in this framework. There is an alternative version of the Black formula that assumes that forward rates are normal, not lognormal, and that will allow interest rates to go negative. Much more on the zero lower bound and negative interest rates below.

2.3 BGM Model

An extension of the Black (1976) model entails specifying that a vector of forward rates $(f(\tau_1, \tau_1+n), f(\tau_2, \tau_2+n)...f(\tau_K, \tau_K+n))'$ follows a multivariate geometric Brownian motion with variance-covariance matrix $V(t) = [v_{ij}(t)]$. This is known as the BGM model (Brace, Gatarek, and Musiela, 1997), or the LIBOR market model. The variance-covariance matrix can in turn be specified to depend on a smaller number of parameters—a common parameterization (Rebonato, 1999) is:

$$v_{ij}(t) = \sigma_i(t)\sigma_j(t)\exp(-\beta|i-j|) \tag{2.9}$$

where

$$\sigma_i(t) = (a(\tau_i - t) + b)\exp(-c(\tau_i - t)) + d \tag{2.10}$$

This constrains $V(t)$ to depend only on five parameters. These parameters can be estimated by minimizing the sum of squared deviations between actual and model-implied interest rate derivatives prices. Having obtained the entire joint risk-neutral distribution of forward rates, the researcher can work out the implied density of many objects of interest, such as the timing of the peak of a tightening cycle. Elliott and Noss (2015) discuss applying the BGM model for such purposes. I will illustrate this use of the model in section 4, below[6].

The BGM model is closely related to the well-known HJM model (Heath, Jarrow, and Morton, 1992). The difference is that Heath, Jarrow, and Morton (1992) consider a law of motion for instantaneous forward rates. Unfortunately, it turns out that lognormal instantaneous forward rates are inconsistent with an absence of arbitrage. For this reason, the BGM model retains the HJM framework but replaces instantaneous forward rates with discretely compounded forward rates, which have the useful by-product of being of more direct relevance to market participants, as actual market rates all have discrete accrual periods. Sandmann and Sondermann (1997), Goldys, Musiela, and Sondermann (2000), Jarrow (2009) and Shreve (2004)[7] all discuss this motivation for using discretely compounded forward rates. Essentially, the BGM model can be thought of as a device that circumvents a technical problem of explosive rates in the HJM model.

2.4 PDFs at a given point in time

Another approach is to collect a set of option prices for a fixed maturity date and to reverse-engineer a pdf for $y_t(n)$ from these prices without making any parametric assumptions. The key insight in doing this is the result of Breeden and Litzenberger (1978) that if $C(K)$ is the price of a call option at a strike price of K on a future interest rate $y_\tau(n)$, then the cumulative distribution function of $y_\tau(n)$ is

$$F(x) = -\frac{1}{P(t,t+\tau)}\frac{\partial C(x)}{\partial K} \tag{2.11}$$

and the probability density function is

$$f(x) = \frac{1}{P(t,t+\tau)}\frac{\partial^2 C(x)}{\partial K^2} \tag{2.12}$$

We do not observe options at every strike price, and so cannot literally compute the derivatives required for these equations. But we can work out the Black-implied

[6] A different approach, that also gives a researcher the joint risk-neutral distribution of future short term interest rates is to calibrate models of short-term interest rates, such as the Hull and White (1990) one-factor model, or the Longstaff and Schwartz (1992) two-factor model, to interest rate derivatives prices. I do not consider this further in the current paper, but Rebonato (2003) and Brigo and Mercurio (2006) give extensive discussion of these methods.

[7] See subsection 10.4.1 of this reference for a textbook treatment.

volatility for each option, interpolate these implied volatilities over an arbitrarily fine grid of points, convert these back into call prices, and then compute the required derivatives numerically. This approach is implemented by Shimko (1993) and Bliss and Panigirtzoglou (2002). Shimko (1993) uses a polynomial in K to smooth the implied volatilities, whereas Bliss and Panigirtzoglou (2002) use a cubic spline. In Monte-Carlo simulations, Bliss and Panigirtzoglou (2002) find that this method is reliable even with quite a small numbers of options.

Another related approach is the local linear regression of Aït-Sahalia and Duarte (2003). Consider a set of L call options at different strike prices at a given point in time for the same underlying asset. Let C_i denote the price of the ith option, with a strike of k_i. We seek to approximate the price of an option at a strike price k' in a neighborhood around k by a locally linear function $\beta_0(k)+\beta_1(k)(k'-k)$. We estimate the coefficients as:

$$\hat{\beta}_0(k), \hat{\beta}_1(k) = \arg \min_{\beta_0(k),\beta_1(k),\beta_2(k)} \Sigma_{i=1}^{L}\{C_i - \beta_0(k) - \beta_1(k)(k_i - k)\}^2 K(\frac{k_i - k}{h})/h, \quad (2.13)$$

where $K(.)$ is a kernel function and h is a bandwidth. There are expressions for the optimal bandwidth in terms of minimizing integrated mean squared error, but it is always wise to look at the smoothed option price to visually judge if the degree of smoothing is appropriate. The first derivative of the call option price with respect to the strike price can then be obtained as $\hat{\beta}_1(k)$ and the second derivative can be obtained as $\hat{\beta}_1'(k)$.

A variant on this approach—discussed in Aït-Sahalia and Duarte (2003)—is to approximate the price of the option by a locally quadratic function. The second derivative can then be recovered directly from the coefficient on the quadratic term. However, inferring the second derivative from equation (2.13) may work better in finite samples (Aït-Sahalia and Duarte, 2003).

2.5 PDFs smoothed over time

In the previous section, we considered constructing a pdf on a given day, or at a particular point in time. This can be appealing because one can then study how a unique event, such as the introduction of some type of forward guidance by the Fed, shifted the pdf. However, because of concerns about measurement error or the absence of options at the required maturity and strike price, it may be preferable to pool options prices over a longer period of time. Such an approach is proposed in the semiparametric methodology of Aït-Sahalia and Lo (1998).

In this approach, observed options prices are converted into Black implied volatilities. Let these options implied volatilities be $\{\sigma_i\}_{i=1}^n$, where n is the total number of options, pooling all call and put options on all trading days, strike prices and expiration dates. Assume that the implied volatility is a function of a state vector, $\sigma(Z_{it})$, where the state vector, Z_{it}, includes the strike price or moneyness of the option, the maturity of the option, and time-varying variables such as the level and slope of the term structure and/or the options implied volatility of interest rates. The implied volatility function can be estimated nonparametrically by kernel smoothing or by a local linear regression. Implied volatilities can then be converted back to call options prices and fitted call options prices can be constructed for any value of the state vector, giving the Q-measure cdf and pdf in equations (2.11) and (2.12). Intuitively, the value of an option on a given day is constructed by smoothing over options with similar strike prices and maturities and over days when the state vector takes on similar values. The advantage of doing this is that the estimation of the state price density can be made much more precise. The disadvantage is that we have to assume that a low dimensional state vector provides a good fit to implied volatilities, though of course this is an assumption that can be checked by seeing how well the fitted implied volatilities match their actual counterparts. Li and Zhao (2009) apply this methodology to LIBOR derivatives; more recently Wright (forthcoming) applies it to a new but thin market on exchange-traded options on Treasury Inflation Protected Securities (TIPS).

3 Physical density forecasts

There is a large literature on forecasting interest rates, with a recent comprehensive summary in Duffee (2013). Most of the emphasis in this literature is on point forecasting. As is well known, nearly all variation in yields reduces to variation in three factors (Litterman and Scheinkman, 1991). These are the first three principal components of yields that can be interpreted as level, slope and curvature, or are equivalently (up to a rotation) the three coefficients in the Nelson and Siegel (1987) model. Most of the point forecasts for interest rates that are used in the literature amount to using a possibly restricted vector autoregression in these three factors to predict future values of the factors, that in turn imply forecasts of future interest rates. Diebold and Li (2006) propose a dynamic Nelson-Siegel model (Nelson and Siegel, 1987), fitting an AR(1) to each of the Nelson-Siegel coefficients. This is a good benchmark for interest rate forecasting, though it should be noted that the record of this and other methods for interest rate forecasting in out-of-sample prediction exercises is sensitive to the period used for forecast evaluation, and the very simplest forecasting model—a random walk forecast—is hard to beat consistently (Duffee, 2013).

In this article, my focus is on density forecasting, not point forecasting. In line with standard terminology, I refer to the physical density forecast as the P-measure density forecast. To construct a non-trivial density forecast (one that is not just a fixed interval around the point forecast), time-varying volatility is needed. Authors studying density forecasts for interest rates include Hong, Li, and Zhao (2004) and Carriero, Clark, and Marcellino (2014). One prominent approach, following Koopman, Mallee, and der Wel (2010), Hautsch and Ou (2012) and Hautsch and Yang (2012) is to construct density forecasts by a dynamic Nelson-Siegel model, but with stochastic volatility. Specifically, suppose that the yield curve is:

$$y_t(n) = \beta_{1t} + \beta_{2t}\frac{1-\exp(-\lambda n)}{\lambda n} + \beta_{3t}[\frac{1-\exp(-\lambda n)}{\lambda n} - \exp(-\lambda n)] \tag{3.1}$$

where λ is fixed, following Diebold and Li (2006). The $\{\beta_{jt}\}_{j=1}^3$ are observed variables that can be inverted from actual yields. Assume that:

$$\beta_{jt} = \phi_{0j} + \phi_{1j}\beta_{jt-1} + \sigma_{jt}\varepsilon_{jt} \tag{3.2}$$

$$\ln(\sigma_{jt}^2) = \omega_{0j} + \omega_{1j}\ln(\sigma_{jt-1}^2) + \kappa_j u_{jt} \tag{3.3}$$

where $\{u_{jt}\}$ and $\{\varepsilon_{jt}\}$ are iid standard normal random variables. Bayesian Markov-Chain Monte-Carlo methods can be applied to equations (3.2) and (3.3), as in Koopman, Mallee, and der Wel (2010), Hautsch and Ou (2012) and Hautsch and Yang (2012). Hence density forecasts can be obtained by stochastic simulation. The model does not explicitly incorporate the zero lower bound, although one can naturally truncate the resulting interest rate forecasts at zero.

The model is already quite a simple and tightly parameterized specification. One could restrict the model further. One possibility is to specify that

$$\phi_{0j} = 0,\ \phi_{1j} = 1\ \forall j. \tag{3.4}$$

This implies that random walk forecasts are used for all the factors, and hence for all yields (as noted above, random walk forecasts are surprisingly competitive). Another possibility is to specify that

$$\omega_{0j} = \omega_{1j} = 0 \ \forall j. \tag{3.5}$$

This implies that the innovation variance in equation (3.2) is constant at κ_j^2. This in turn means that any density forecast is just a fixed interval around the point forecast.

Here, as in most of the interest rate forecasting literature, I have considered using just yield curve variables for forecasting. As emphasized by Duffee (2013), barring unusual circumstances, if a factor is relevant for predicting future interest rates, then it should be reflected in bond yields. Nevertheless, many researchers have used macroeconomic variables for interest rate forecasting, especially in point forecasting, but also in density forecasting. For example Clark (2011) considers density forecasts for the federal funds rate from a vector autoregression with stochastic volatility in GDP growth, unemployment, inflation and the funds rate.

Whereas Q-measure dynamics are well identified from derivatives prices, P-measure dynamics are hard to pin down. With any of the specifications discussed above, the researcher is basing a density forecast on the historical time series behavior of interest rates, and that is associated with various econometric problems, including downward bias in persistence estimates, and even more importantly a very flat likelihood. It takes a long sample without structural breaks to estimate mean reversion of persistent data precisely. Motivated by this problem, in the context of point forecasting, Kim and Orphanides (2012) propose incorporating survey forecasts of future interest rates as a way of sharpening identification under the P-measure. They treat surveys as noisy measures of agents' physical expectations, and show that this helps a great deal with identification, and often prevents the model from giving unreasonable assessments of physical expectations. This remains true even allowing for considerable measurement error in the surveys[8]. The benefits of using surveys are especially great at or near the zero lower bound, as there is good reason to think that there have been structural breaks. There is virtually no historical experience at such low interest rates to guide a time series based model estimation procedure.

The Markets Group of the Federal Reserve Bank of New York conducts a survey of Primary Dealers ahead of each FOMC meeting, and since 2011 the results have been

[8] Chun (2011) also considers the use of survey forecasts in term structure analysis.

made public. This survey includes density forecasts for future interest rates. Respondents assign probabilities to the federal funds rate falling into discrete bins, and these probabilities are then averaged over respondents. In light of the great difficulties in constructing physical density forecasts from time series econometric methods, I think that it is reasonable to view these survey density forecasts as approximating the P-measure density forecasts of investors. At a minimum, they can be useful information in calibrating physical density forecasts. The New York Fed survey is the only survey *density* forecast for interest rates that I am aware of. The questions are very specific and change from survey to survey, and it is designed to answer questions of current interest to the FOMC rather than for longer term research purposes. Nonetheless, it is a unique and valuable resource.

3.1 The pricing kernel

Armed with the P- and Q-measure pdfs, the pricing kernel can be worked out as the ratio of the Q-measure pdf to the P-measure pdf. Given that the data at hand are derivatives prices plus the historical behavior of interest rates, it seems most natural to me to compute the Q-measure pdfs (from derivatives prices) and the P-measure pdfs (typically from the time series properties of interest rates), and then to work out the pricing kernel as a by-product. That also enables us to exploit the fact that Q-measure pdfs are very well identified and can be estimated with quite flexible specifications whereas P-measure pdfs are poorly identified and tightly parameterized models may be most appropriate for them.

Still, this is not the only possible way to proceed. Rosenberg and Engle (2002) estimate physical pdfs and then parameterize a pricing kernel, estimating the parameters of the pricing kernel so as to match options prices. Effectively, this identifies the P-measure pdf and the pricing kernel, and then works out the Q-measure pdf as a by-product.

3.2 Term structure models

Affine and quadratic term structure models jointly specify the physical and risk-neutral processes for interest rates and the pricing kernel, and can also be used to construct P- and Q-measure pdfs for interest rates. Egorov, Hong, and Li (2006) consider the use of affine term structure models for interest rate density forecasting, and compare the resulting forecasts with homoskedastic random walk forecasts. They find that while the random walk forecast gives more accurate forecasts of the conditional mean, the affine term structure model gives better density forecasts, giving some evidence for the importance of modeling volatility.

Term structure models are typically fitted to yields data alone, and can fit conditional means quite well, but have more trouble in fitting higher moments (Collin-Dufresne, Goldstein, and Jones, 2009; Jacobs and Karoui, 2009). The ability to fit higher moments appears to be sensitive to the sample period. Several papers have documented the importance of allowing for additional volatility factors driving interest rate derivatives prices that cannot be inverted from yields (Andersen and Benzoni, 2010; Collin-Dufresne and Goldstein, 2002). This potential phenomenon is generally referred to as *unspanned stochastic volatility*.

There have been several recent papers fitting term structure models to interest rates and derivatives jointly. All of them involve both factors in the conditional mean (effectively level, slope and curvature) and additional volatility factors. Allowing for these additional factors, the models are able to fit both interest rates and derivatives prices well. Heidari and Wu (2009) and Cieslak and Povala (2016) consider affine term structure models with conditional mean and volatility factors—Cieslak and Povala also use high-frequency data to obtain precise estimates of the physical second moments of yields. Filipović, Larsson, and Trolle (forthcoming) propose a model in which the state price density is a linear function of a state vector and show that it implies that bond yields are ratios of linear functions of those factors. While these papers have generally focused on obtaining a good in-sample fit, their models might also be useful for density forecasting. It has generally been seen as desirable for a term structure model to impose the non-negativity of interest rates–the model of Filipović, Larsson and Trolle imposes non-negative rates[9] whereas Cieslak and Povala do not. But the case is maybe less clear now than it appeared in the past, as negative bond yields are now common in many countries outside the US. More on this below.

[9] Filipović, Larsson, and Trolle (forthcoming) find that their model fits swap and swaptions quotes well even during the ZLB period.

4 Applications

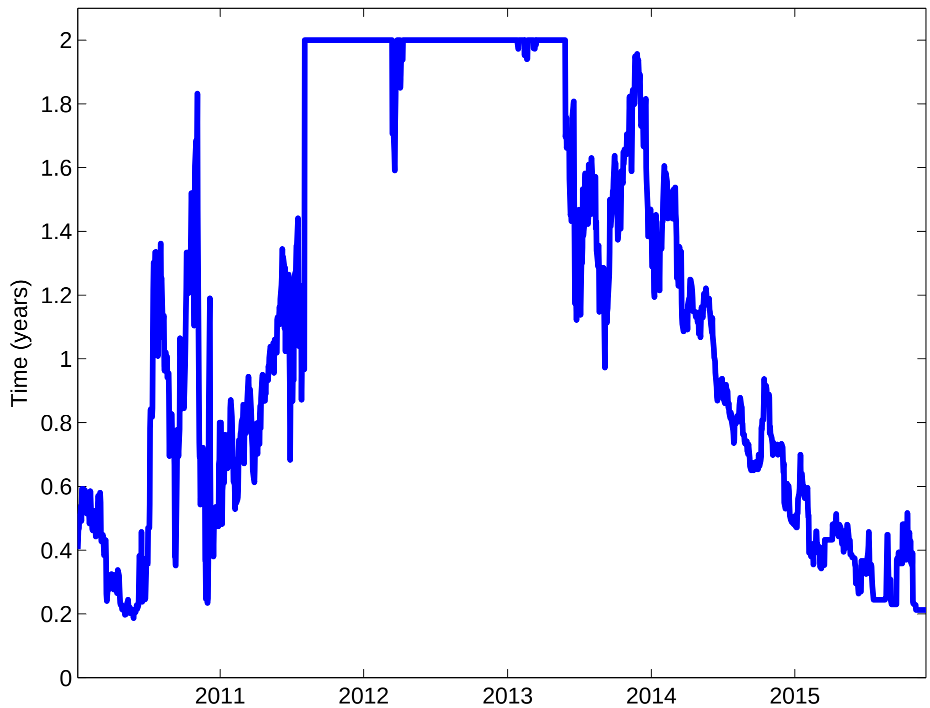
In this section, I turn to some illustrative applications of the methods described above, with special attention to forward guidance and issues raised by the ZLB.

4.1 Options-implied PDFs for short-term rates

I first consider Q-measure density forecasts for three-month interest rates implied by Eurodollar options quotes. Eurodollar options are available expiring each March, June, September and December. For each day from January 2010 to May 2016, I use settlement quotes on call options at up to 41 different strike prices. I linearly interpolated these to constant maturities, and then obtained cdfs/pdfs by local linear regression (equation (2.13)). Eurodollar options are based on LIBOR quotes, that at times incorporate substantial liquidity premia and/or credit risk premia (Hull and White, 2013). As a simple attempt to correct for this, I shifted the densities each day by the amount of the LIBOR-OIS spread, and so they can be thought of as referring to three-month OIS rates, that can in turn be thought of as pure forward discount rates. I take the cdf at 30 basis points (equation (2.11)) as the probability of being at the ZLB. When the US was at the ZLB from 2008 to 2015, the target for the federal funds rate was a range from 0 to 25 basis points, and there was little serious discussion of lowering the funds rate during this period.

As a way to condense the information in these pdfs/cdfs, I use them to compute the risk-neutral median time to liftoff, defined as the shortest horizon at which the cdf of three-month OIS rates at 30 basis points is below 0.5. This is shown in Figure 1, from January 2010 to November 2015. It is a measure of how much scope the Fed had to ease financial conditions via forward guidance, separate from using negative rates or large-scale asset purchases. The time is censored at two years (if the cdf at the two year horizon is above 0.5), because of the sparsity of options quotes at longer maturities. In 2010, the time to liftoff was very short, at under 6 months. By 2012, it had risen to at least 2 years. This is probably about the limit of what forward guidance can credibly achieve, short of an economy being widely perceived to be mired in an intractable slump. Later the median time to liftoff came back down as the Fed signaled that the normalization of monetary policy would begin soon.

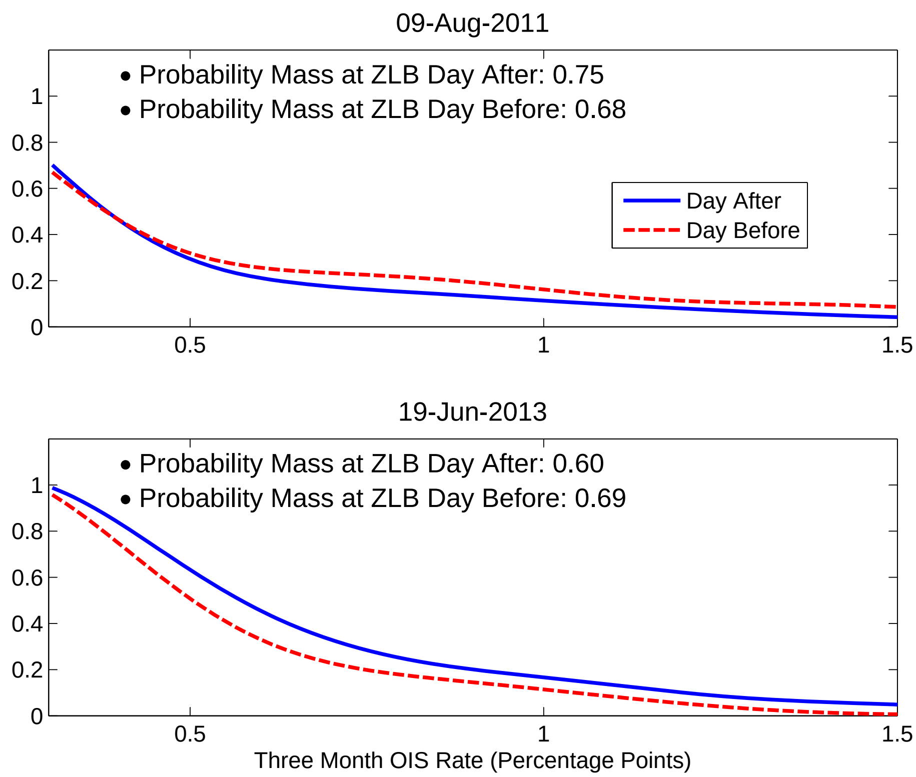
Figure 1: Risk Neutral median time to liftoff



NOTE: This figure plots the time in years until the (forward) risk neutral cdf for three month OIS rates evaluated at 30 basis points first falls to 0.5, on each day. The time is censored at two years—if the cdf at the two year horizon is above 0.5, then the time to liftoff is reported as two years. These cdfs are constructed by local linear regressions using Eurodollar options, as described in the text. The densities are shifted by the amount of the LIBOR-OIS spread, so as to be interpretable as pdfs for three-month OIS rates.

For a more granular look at a couple of key events, Figure 2 plots the pdfs at the one-year horizon for the days before and after August 9, 2011 and June 19, 2013 (panels A and B, respectively). August 9, 2011 was the day that the FOMC announced that it expected to keep rates exceptionally low at least through the middle of 2013, substantially sharpening its forward guidance. June 19, 2013 was the date that the FOMC announced a plan to taper and end its large-scale asset purchases sooner than had been expected. Figure 2 allows us to see the effects of these announcements on the entire risk-neutral pdf of interest rates. Both panels show the pdf for interest rates above 30 basis points, and the point mass associated with the ZLB (i.e. the cdf at 30 basis points). I represent the densities in this way, because the densities put very high probability on interest rates just below 30 basis points, clearly corresponding to the target federal funds rate remaining in the range from 0 to 25 basis points. In panel A, it can be seen that the announcement on August 9, 2011 increased the probability of being at the ZLB one year hence from 68% to 75%, and the probability associated with a substantial tightening of monetary policy was correspondingly marked down. The forward guidance was therefore quite effective. In panel B, it can be seen that the announcement on June 19, 2013 reduced the probability of being at the ZLB

Figure 2: Risk Neutral PDFs for three month rates one year hence around selected FOMC meetings



NOTE: This figure plots the (forward) risk neutral pdfs for three month OIS rates at the one-year-horizon as of the days before and after two FOMC meetings. These are constructed by local linear regressions using Eurodollar options, as described in the text. The densities are shifted by the amount of the LIBOR-OIS spread, so as to be interpretable as pdfs for three-month OIS rates. The figures show the continuous densities above 30 basis points, along with the value of the cumulative distribution function at 30 basis points (probability mass at the ZLB).

one year hence from 69% to 60%. The tapering announcement evidently led market participants to increase their odds of liftoff from the ZLB within a year.

This exercise illustrates that we can study the effects of specific announcements—including rather unique one-off announcements as those considered in Figure 1—on risk neutral pdfs for interest rates. In doing this, it is important that entirely separate pdfs are constructed right before and after the event. Methodologies that pool data over a long period in forming pdfs, as described in subsection 2.5, are not well suited to examining the effect of specific events.[10]

4.2 Shadow interest rates

Clearly the ZLB does not just affect the overnight interest rate; it also affects yields further out the term structure. Swanson and Williams (2014) look at the sensitivity of the term structure of yields to economic announcements to see how far out the term structure the ZLB reaches. Options provide an alternative approach for doing this.

Assume that the three-month OIS interest rate at time τ is $\max(\underline{s}, s_\tau)$—as in the Black (1995) model—where s_τ, the shadow rate for time τ is $N(\mu, \omega^2)$ and $\underline{s}$ is the lower bound (taken as 30 basis points). Under the Q-measure, using results on the mean of a truncated normal, the expectation of three-month interest rate will be:

$$\mu + \Phi(\alpha)(\underline{s} - \mu) + \phi(\alpha) \tag{4.1}$$

where $\alpha = \frac{\underline{s}-\mu}{\omega}$.This will be equal to the futures rate, F. Meanwhile, the probability of three-month rates being at the ZLB is $P_Z = \Phi(\alpha)$ and this can be recovered from options quotes. Rearranging we can solve for μ as:

$$\mu = \frac{F - \underline{s}P_Z - \phi(\Phi^{-1}(P_Z))}{1 - P_Z} \tag{4.2}$$

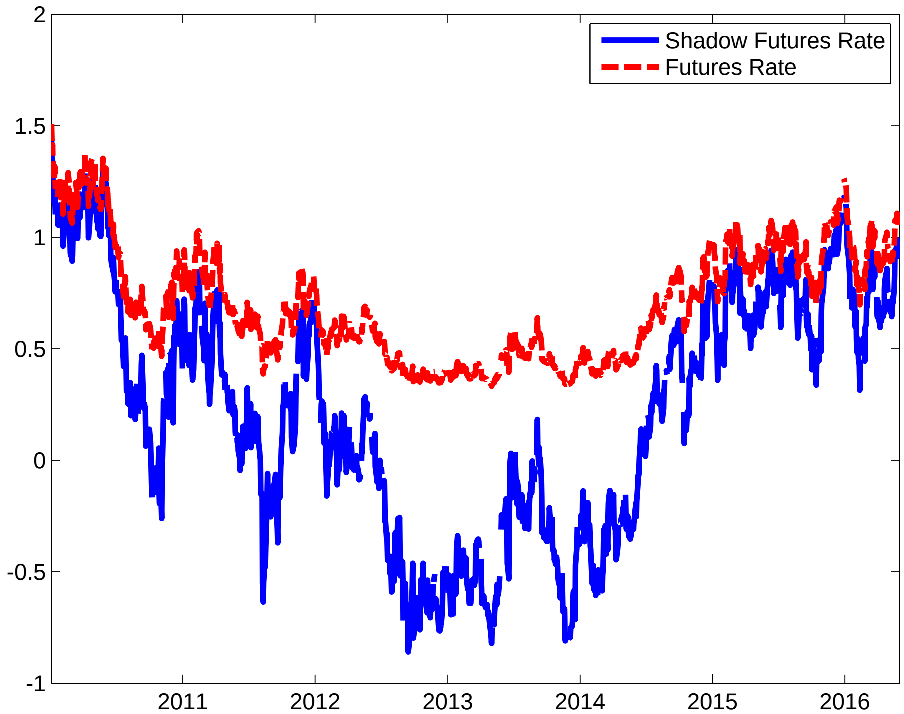
The parameter μ tells us what the futures rate for time τ would be, in the absence of the ZLB. I call this a shadow futures rate. It is a different concept of a shadow rate than that conventionally employed in the ZLB literature. Black (1976) defines the shadow rate as what the current level of the short rate would be in the absence

[10] A "movie" showing the pdf at the one-year horizon on every day from January 2010 to May 2016 is available from the website http://www.econ2.jhu.edu/People/Wright/edmovie.mov.

of a non-negativity constraint. Wu and Xia (2016) show how to reverse-engineer this shadow rate from how close short- and intermediate-maturity yields are to zero. Instead I am defining a shadow futures rate as what the *futures* rate would be in the absence of a non-negativity constraint, and identifying this from options prices alone.

In Figure 3, I plot the futures rate and the shadow futures rate at the one-year horizon. In 2010 and early 2011, the two were close as the probability of an early exit from the ZLB was seen as high. From mid-2011 to mid-2014, the shadow futures rate was well below the actual futures rate, as the ZLB was binding further out the term structure. After mid-2014, the shadow and actual futures rates converged once again, as liftoff came near. These findings are consistent with those of Swanson and Williams (2014), though obtained in a different and perhaps more direct way.

Figure 3: Actual and shadow three-month futures rates one year hence



NOTE: The actual futures rate is a one-year ahead rate at constant maturity formed by linearly interpolating between Eurodollar futures quotes, less the current LIBOR-OIS spread. The shadow rate is formed from equation (4.2) and represents what the futures rate would have been in the absence of the ZLB, as implied by options quotes.

4.3 Calibration of the BGM model

To illustrate the use of the BGM model (discussed in subsection 2.3), I took at-the-money swaptions prices on May 31, 2016, from Bloomberg, at 1, 2, 3, 5, 7 and 10 year exercise dates and 1, 2, 3, 5, 7 and 10 year underlying swap maturities, and calibrated the BGM model parameterized as in equations (2.9) and (2.10), so as to minimize the sum of squared swaptions pricing errors. Payoffs were discounted using the zero-coupon yield curve[11] of Gürkaynak, Sack, and Wright (2007)[12]. The calibrated parameters (a, b, c, d and β in the notation of these equations) then give a complete characterization of the risk-neutral evolution of future interest rates, which allow questions to be answered that cannot be addressed by looking at a single maturity point in isolation.

As an illustration, I simulated the calibrated model and computed the Q-measure time until the first peak in one-year interest rates. Table 1 shows these probabilities. Interestingly, the implied probability of a peak in one-year rates within the next two years was nearly 70 percent, meaning that options implied a better-than-even chance of the current Fed tightening cycle ending before mid-2018.

Table 1: Risk neutral probabilities of time to first peak in one-year yields as of May 31, 2016

Time to first peak	Probability (%)
Less than 1 year	44.8
1-2 years	25.7
2-3 years	16.1
3-4 years	6.6
4-5 years	4.9
More than 5 years	1.9

Notes: This table shows the probability density function of the time to the first peak in one-year yields from simulations of a BGM model calibrated to swaptions quotes as of May 31, 2016.

[11] It used to be market convention to value derivatives using LIBOR rates for discounting. But this has changed since the financial crisis. Other rates—most often OIS rates—are now sometimes preferred instead (Hull and White, 2013).

[12] This uses the Svensson (1994) parameterization of the yield curve. See Jarrow (2014) for a discussion of zero coupon and forward yield curve fitting methods.

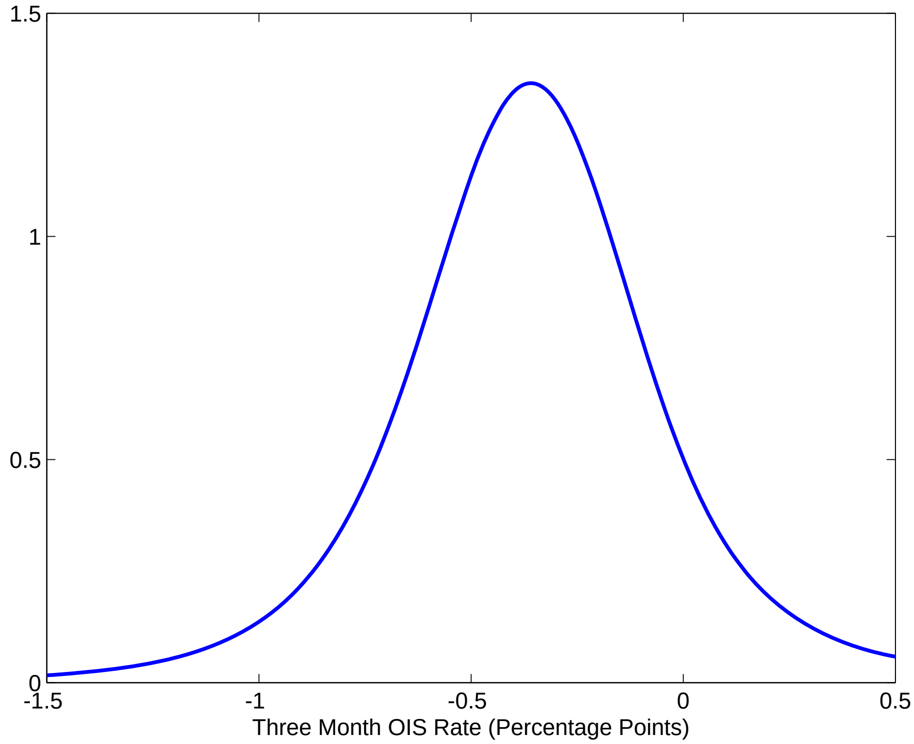
4.4 Options implied pdfs and negative rates

In the US, the ZLB has consistently kept the federal funds rate above zero, and has prevented three-month LIBOR from falling much below 50 basis points. Occasionally, negative yields were observed in the secondary Treasury bill market, but these were very isolated cases. The approach in the Black model, or in most papers in zero lower bound literature (including Black (1995), Krippner (2014), Wu and Xia (2016) and Kim and Singleton (2012)), is simply to rule out negative interest rates. This seems appropriate in the US context. Federal Reserve Chair Janet Yellen stated in Congress (February 11, 2016) that it was unclear if the FOMC had the legal authority to set negative interest rates. That alone makes it quite unlikely that negative interest rates will be used in the US.

But other countries grappling with overvalued currencies (especially Switzerland) and/or sustained economic weakness (especially the euro area and Japan) have resorted to negative interest rates. This has come as something of a surprise to many observers. Nonetheless, to date, fears that this would lead to warehousing of large amounts of cash have not proved correct (Rognlie, 2016), although there is surely some tipping point that would lead the financial system to develop a new infrastructure to hold enormous cash balances. In these countries, the ZLB seems better thought of a soft constraint. Options can then be used just as before to measure the market assessment of how far rates might go into negative territory.

Euribor options are the euro-area analog of Eurodollar options. Although most of the literature on options-implied interest rate densities is applied to the US, there have been some applications to the euro area (Andersen and Wagener, 2002; Ivanova and Gutiérrez, 2014). Because of its size, and negative rates, the euro area is of special significance at present. Consequently, Figure 4 plots the Q-measure pdf for euro-area three-month interest rates in one year's time, as of May 31, 2016. These were obtained in the same way as for the US, again with an adjustment for the LIBOR-OIS spread, though that was very modest. On May 31, 2016, the three-month euro OIS rate was -34 basis points. The pdf implies low odds rates moving back above zero, but substantial odds of rates moving somewhat further into negative territory by mid-2017.

Figure 4: Risk Neutral PDF for euro-area three month rates one year hence as of May 31, 2016



NOTE: This figure plots the (forward) risk neutral pdfs for three month OIS rates at the one-year-horizon for the euro area as of May 31, 2016. These are constructed by local linear regressions using Euribor options, as described in the text. The density is shifted by the amount of the euro LIBOR-OIS spread, so as to be interpretable as a pdf for three-month OIS rates.

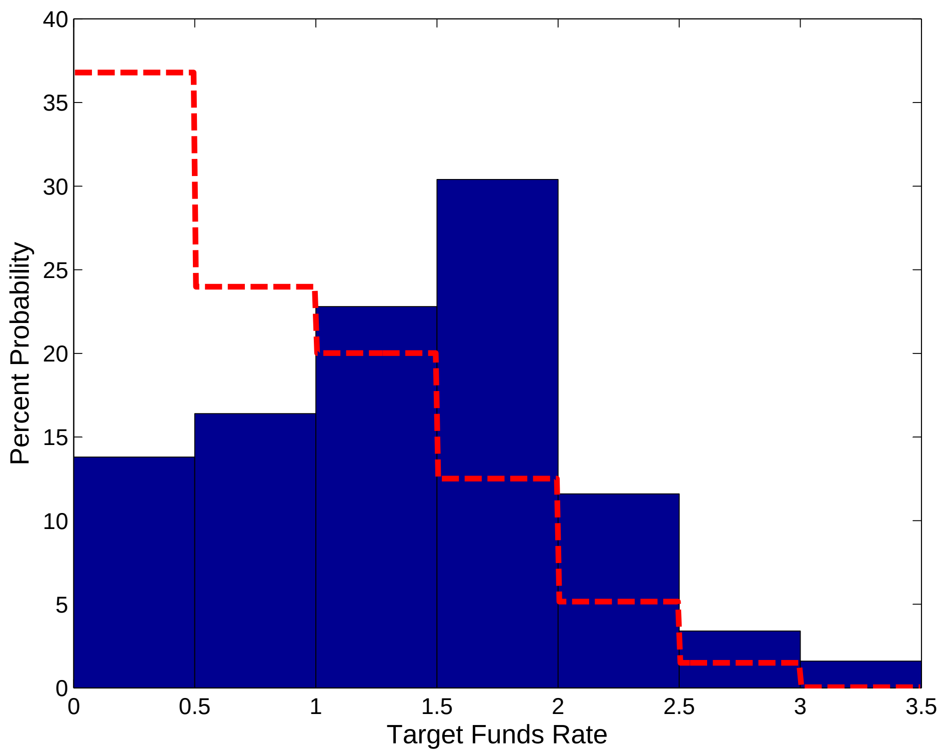
4.5 Physical interest rate density forecasts

Generally time series models are used to construct P-measure interest rate forecasts. However, in this paper, owing to the difficulties discussed in section 3 pertaining to time series models of interest rates, especially in the current environment, I instead use the Federal Reserve Bank of New York Primary Dealer survey. Figure 5 plots the pdf for the target federal funds rate at the end of 2017 from the April 18, 2016 Primary Dealer survey[13]. The Q-measure pdf for the OIS rate over a three-month

[13]The survey asks respondents for the pdf conditional on moving to the ZLB at some point in 2016-2018, the pdf conditional on *not* moving to the ZLB at any point in 2016-2018, and the probability of moving to the ZLB during that period. Figure 5 plots the implied unconditional pdfs.

window centered around the end of 2017, as of the survey date, is also shown in the figure. This was obtained from Eurodollar options, as discussed in subsection 4.1, and represents an approximate Q-measure pdf for the target funds rate at the end of 2017. This makes the P- and Q-measure pdfs as directly comparable as possible. Another alternative would be to use Federal Funds futures options, which settle to the average

Figure 5: Physical (Survey) and Risk-Neutral PDFs for the end-2017 funds rate as of April 18, 2016



NOTE: This figure plots the survey probabilities for the target federal funds rate at the end of 2016 from the Federal Reserve Bank of New York Primary Dealer Survey of April 18, 2016. Probabilities are put into bins that are 50 basis points wide, with the left-most bin referring to interest rates of less than 50 basis points and the right-most bin referring to interest rates of greater than 3 percentage points. The survey asks respondents for probabilities conditional on a return to the ZLB in 2016-2018, conditional on no return to the ZLB in 2016-2018, and the probability of a return to the ZLB in that period. The probabilities plotted as bars in this figure are the implied unconditional probabilities. In addition, the figure shows the corresponding risk neutral pdfs for three-month OIS rates on November 15, 2017 (at the 18 month horizon—red dashed lines). These are constructed by local linear regressions using Eurodollar options, as described in the text. The density is shifted by the amount of the LIBOR-OIS spread, so as to be interpretable as a pdf for the federal funds rate around the end of 2017.

effective federal funds rate for a particular month. But these are very illiquid, except at the shortest maturities.

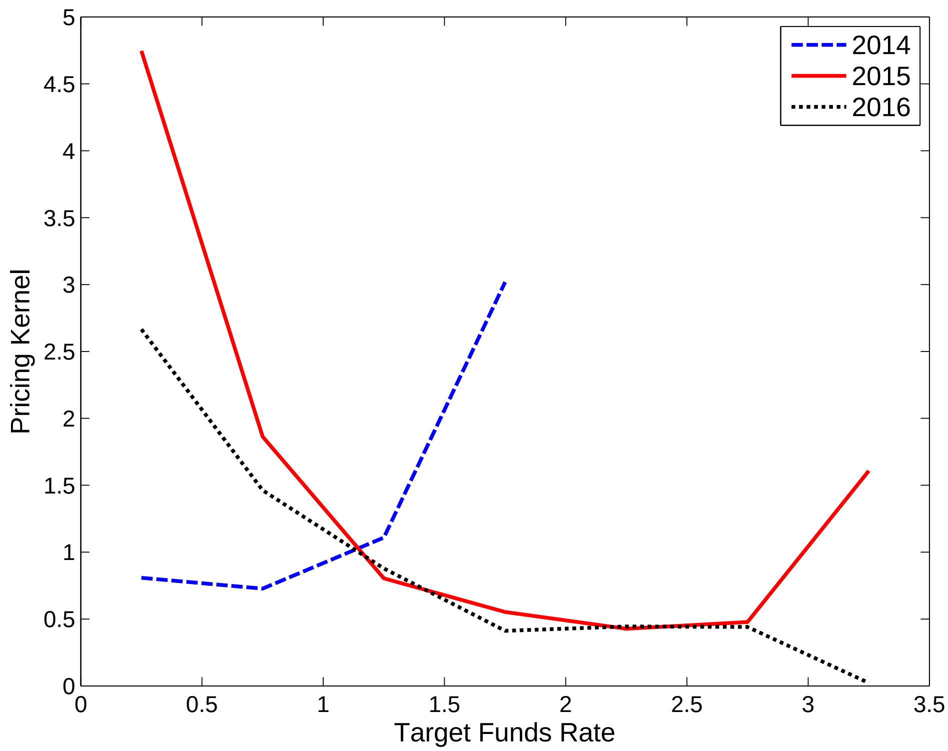
A useful spinoff of the Primary Dealer survey is that it can be used to compute the *mean* forecast of the federal funds rate, whereas most surveys are ambiguous as to whether they elicit the mean, median or modal forecast (Engelberg, Manski, and Williams, 2009). In Figure 5, the mean survey forecast for the funds rate at the end of 2017 was 1.38 percent whereas the corresponding modal forecast was 1.5 to 2 percent.

4.6 Pricing Kernel

In Figure 5, it can be seen that the survey reported 14% odds of the federal funds rate being less than 50 basis points, with the rest of the probability mass being distinctly away from the ZLB. Meanwhile, the Q-measure pdf derived from Eurodollar options put much more mass on very low interest rates. The empirical pricing kernel—the ratio of Q- to P-measure pdfs—is thus downward sloping. Figure 6 plots the pricing kernel constructed in this way on April 18, 2016, and also on dates in April 2014 and April 2015, likewise constructed from the Primary Dealer survey for interest rates at the end of the following year, coupled with Eurodollar options quotes.

Most researchers have found pricing kernels to be U-shaped in interest rates. Li and Zhao (2009) computed the pricing kernel by comparing P- and Q-measure pdfs using interest rate caps from 2000 to 2004. They found that it was U-shaped in interest rates—both very high and very low interest rates were high marginal utility states of the world. Ivanova and Gutiérrez (2014) also found U-shaped pricing kernels using Euribor options data from 2006 to 2012. In Figure 6, I also find something of a U-shaped pattern in the empirical pricing kernel in 2014 and 2015. But in 2016, I find that investors are willing to pay a premium to hedge against the low interest rate states of the world, but not the high interest rate states. This may be reasonable, as the predominant macroeconomic worry today is slow growth, deflation and/or financial instability—circumstances that will be associated with low levels of the federal funds rate (Campbell, Sunderam, and Viceira, 2009). It is also entirely consistent with the finding that term premia are presently negative in the models of Kim and Wright (2005) and Adrian, Crump, and Moench (2013), although those papers focus on the Treasury term premium, not the term premium in short-term money market futures[14]. The differences between my results and those of Li and Zhao and Ivanova and Gutiérez arise because I use more recent data and take the P-measure pdf from surveys, not from time series methods. Ivanova and Gutiérez use six-year rolling windows of actual interest rates to form the P-measure pdf, and while that might be reasonable in normal times, I prefer to use survey evidence given the unusual recent behavior of interest rates.

Figure 6: Empirical Pricing Kernels



NOTE: This figure plots the empirical pricing kernel constructed as the ratio of the risk-neutral pdfs for three-month OIS rates at the 18 month horizon to the survey probabilities for the target federal funds rate at the end of the following year from the Federal Reserve Bank of New York Primary Dealer Surveys of April 22, 2014, April 20, 2015 and April 18, 2016. Survey probabilities are put into bins that are 50 basis points wide. Risk-neutral pdfs are constructed by local linear regressions using Eurodollar options, as described in the text. The density is shifted by the amount of the LIBOR-OIS spread, so as to be interpretable as a pdf for the federal funds rate around the end of following year.

[14] At the time of writing, the term premium in short-term interest rate futures has to be negative in order to give a reasonable expected path of monetary policy.

5 Conclusion

With the wealth of interest rate derivatives that trade in deep and liquid markets, researchers can identify the state price density for future interest rates with great precision and minimal assumptions. The physical density is much harder to recover, but progress can be made on that front too, via time series econometric methods and/or surveys. Depending on the purpose, researchers may be most interested in either the physical or risk-neutral density (Feldman, Heinecke, Kocherlakota, Schulhofer-Wohl, and Tallarini, 2015)

In this paper, I have reviewed methods for interest rate density forecasting under both risk-neutral and physical measures. I have shown some illustrative applications, with particular focus on issues relating to the zero lower bound. The main goal of these applications is to show that forward looking estimates of interest rate distributions are important not just to finance practitioners, but also in macroeconomics and monetary economics, where these methods remain regrettably underutilized.

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