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Probabilistic sophistication without completeness☆

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ABSTRACT

This is a study of probabilistically sophisticated choice behavior when the preference relation is incomplete. Invoking the analytical framework of Anscombe and Aumann (1963) and building on the work of Machina and Schmeidler (1995), the paper provides an axiomatic characterization of the general multi-prior multi-utility probabilistically sophisticated representation. In addition, the paper examines the axiomatic foundations for two special cases: complete beliefs and complete tastes. In the former case, the incompleteness is due to ambiguous tastes and in latter case it is due to ambiguous beliefs.

1. Introduction

Choice-based definition of subjective probabilities presumes that, when called upon to decide among courses of action whose consequences are not known in advance, decision makers form of beliefs about the likely realization of the consequences, and that these beliefs are quantifiable by probabilities. Because the beliefs are personal, their representation is dubbed subjective probabilities. Borel (1924), Ramsey (1931) and de Finetti (1937) were first to propose the key idea that subjective probabilities may be inferred from the odds a decision maker is willing to offer when betting on events or the truth of propositions. This idea found its ultimate expression in the seminal works of Savage (1954) and Anscombe and Aumann (1963). A common feature of these works is that the subjective probabilities are defined in the context of expected utility theory. Consequently, these works confound the definition of subjective probabilities with the hypothesis that individual choice among uncertain prospects is representable by a functional that is linear in the probabilities. However, the representation of a decision maker's beliefs by subjective probabilities and the notion of expected utility maximizing choice behavior are two separate ideas.

Machina and Schmeidler (1992, 1995) severed this connection by proposing a model, dubbed probabilistic sophistication, in which choice-based subjective probabilities are defined without requiring that the decision maker's preferences respect the strictures of expected utility theory. According to Machina and Schmeidler subjective probabilities transform acts (that is, random variables on a state space that take their values in the set of consequences) into lotteries (that is, the corresponding probability distributions on the set of consequences) and preferences are represented by a utility function over the set of lotteries.

A central tenet of both the expected utility models and the probabilistically sophisticated models is that all alternative courses of action are comparable. That this presumption is not tenable as a general depiction of real-life decision making was recognized by von Neumann and Morgenstern who wrote "It is conceivable – and may even in a way be more realistic – to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable" (von Neumann and Morgenstern, 1947, p. 19). Aumann (1962) finds universal comparability not only an inaccurate description of real-life decision making but also lacking normative appeal. In his words, "Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from a normative viewpoint" (Aumann, 1962, p. 446). Empirically, the main manifestation of incomplete preferences is indecisiveness or inertia.

Considering the restrictive nature of the completeness requirement, the objectives of this paper are to examine the implications of relaxing the completeness axiom in Machina and Schmeidler's theory of probabilistically sophistication choice, and to study the representations of ambiguous beliefs and tastes in this model. More specifically, invoking the analytical framework of Anscombe and Aumann (1963), I explore conditions under which incomplete preference relations admit a multi-utility

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multi-prior in the probabilistically sophisticated representation.[1] In addition, I explore the conditions that characterize two special cases: Knightian uncertainty and single-prior multi-utility representation. The former case, first explored by Bewley (2002) in the context of expected utility theory, attributes the incompleteness to the decision maker's ambiguous beliefs and the latter, explored by Shapley and Baucells (1998) and Dubra et al. (2004), to his ambiguous tastes.

The main analytical difficulty introduced by the incompleteness of the preference relation is due to the non-transitivity of the incomparability relation. When the preference relation is complete, indecisiveness arises only when the decision maker is indifferent among the alternatives under consideration. In Machina and Schmeidler (1995) the indifference is an equivalence relation, hence it is transitive. They exploit the transitivity of indifference to link general Anscombe–Aumann acts to constant acts that are convex combinations of the state-contingent payoffs of the original acts. This link is severed when the incomparability relation is non-transitive. The loss of transitivity requires that the preference structure is enhanced by the introduction of two new axioms dubbed replacement acyclicity and constant-act comparability. Replacement acyclicity requires that the incomparability relation restricted to replacement paths be acyclic. Constant-act comparability requires that one act be strictly preferred over another if and only if the induced constant acts that are non-comparable to the former are strictly preferred to those induced by the latter using the same reduction process.

2. The model

2.1. The analytical framework

Let $S = \{s_1 ..., s_n\}$ be a finite set of *states* and X a set of *outcomes*. Subsets of S are *events.* Denote by ΔX the set of simple probability distributions on X. Elements of ΔX are *lotteries*. Let H be the set of mappings on S to ΔX. Elements of H are *acts* defined by Anscombe and Aumann (1963). Constant acts are identified with elements of ΔX, hence, $\Delta X \subset H$. Let $\succ$ be a binary relation on H. The relation $\succ$ is bounded on X if there are $\bar{x}$ and $\underline{x}$ in X such that $\delta^{\bar{x}} \succ \delta^{x} \succ \delta^{\underline{x}}$, for all $x \in X\backslash\{\bar{x}, \underline{x}\}$, where $\delta^x \in \Delta X$ is the degenerate lottery that assigns the unit probability mass to x. I assume throughout that $\succ$ is bounded.

2.2. The axiomatic structure

The following axioms depict the preference structure.

(A.1) (**Strict partial order**) The strict preference relation $\succ$ is transitive and irreflexive.

The binary relation $\succ$ on H that satisfies (A.1) is referred to it as *strict preference relation*. Define the *incomparability relation*, $\asymp$ on H as follows: For all $f, g \in H, f \asymp g$ if $\neg(f \succ g)$ and $\neg(g \succ f)$. Clearly, $\asymp$ is symmetric and, since $\succ$ is irreflexive, it is reflexive but is not necessarily transitive.

(A.2) (**Archimedean**) For all $f, g, h \in H$, if $f \succ g$ and $g \succ h$ then there exist $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1 - \alpha) h \succ g \succ \beta f + (1 - \beta) h$.

[1] In SEU theory this issue was explored in Seidenfeld et al. (1995), Nau (2006) and Galaabaatar and Karni (2013).

The statement of the next two axioms invokes the following additional notations and definitions. For every event E and $f, g \in H$, let $f_E g \in H$ be the act that agrees with f on E and with g on $S\backslash E$. An event E is *null* if $\neg(\delta^{\bar{x}}_E f \succ \delta^{\underline{x}}_E f)$, for all $f \in H$, and is *nonnull* otherwise. Following Machina and Schmeidler (1995) the lottery p *is said to dominates the lottery qaccording to first-order stochastic dominance*, denoted $p >^1 q$, if $\Sigma_{\{z|z \succ x\}} p(z) \geq \Sigma_{\{z|z \succ x\}} q(z)$, for all $x \in X$ with strict inequality for some $x \in X$. The next axiom requires that first-order stochastically dominating lotteries be preferred.[2]

(A.3) (**Monotonicity**) For all $p, q \in \Delta X$, if $p >^1 q$ then $p_E h \succ q_E h$, for all nonnull $E \subset S$ and all $h \in H$.

The next axiom is a reformulation of the Horse/Roulette Replacement axiom of Machina and Schmeidler (1995) replacing the indifference with the incomparability relation. The idea captured by the Horse/Roulette Replacement axiom is that one can use a probability mixture of the lottery payoffs p and q of the act $p_{E_i}\left(q_{E_j} h\right)$ to define a payoff contingent on the event $E_i \cup E_j$ as an estimate the probabilities of the disjoint events E_i and E_j conditional on their union. The replacement axiom below extends this idea to estimate the sets of probabilities that represent the decision maker's ambiguous beliefs about the likely realization of these events conditional on their union. Formally,

(A.4) (**Replacement**) For every finite partition $(E_1, \ldots, E_n)$ of S, if

$$\delta^{\bar{x}}_{E_i}\left(\delta^{\underline{x}}_{E_j}\delta^{\underline{x}}\right) \asymp \left(\alpha\delta^{\bar{x}} + (1-\alpha)\,\delta^{\underline{x}}\right)_{E_i \cup E_j} \delta^{\underline{x}}$$

for some $\alpha \in [0, 1]$ and pair of events E_i and E_j, then

$$p_{E_i}\left(q_{E_j} h\right) \asymp (\alpha p + (1-\alpha)\,q)_{E_i \cup E_j}\, h$$

for all $p, q \in \Delta X$ and $h \in H$.

The Horse/Roulette Replacement axiom of Machina and Schmeidler (1995) invokes the transitivity of the indifference relation. Since the incomparability relation is non-transitive their chain of replacements by which an act is reduced to an equivalent constant act does not apply. The following axiom replaces the transitivity of indifference with acyclicity of the incomparability relation that is required to hold solely along replacement paths.[3] This requirement assures that the set of probabilities representing the decision maker's ambiguous beliefs is consistently maintained when the event-contingent payoffs of acts are replaced by their corresponding mixtures conditional on the unions of the underlaying events. Formally,

(A.5) (**Replacement acyclicity**) For all $f \in H$, if there are $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ such that

$$\begin{aligned} & f(s_1)_{\{s_1\}}\left(f(s_2)_{\{s_2\}} f\right) \asymp (\alpha_1 f(s_1) + (1-\alpha_1) f(s_2))_{\{s_1,s_2\}} f \asymp \\ & \asymp \alpha_2 (\alpha_1 f(s_1) + (1-\alpha_1) f(s_2)) \\ & \quad + (1-\alpha_2) f(s_3)_{\{s_1,s_2,s_3\}} f \asymp \\ & \alpha_3 (\alpha_2 (\alpha_1 f(s_1) + (1-\alpha_1) f(s_2)) + (1-\alpha_2) f(s_3)) \\ & \quad + (1-\alpha_3) f(s_4)_{\{s_1,s_2,s_3,s_4\}} f \asymp \ldots \asymp \sum_{i=1}^{n} \tau_i p_i \end{aligned}$$

[2] The same property, implied by their adoption of Savage's P3, is a tacit aspect of Machina and Schmeidler (1992). Grant (1995) characterized probabilistically sophisticated preferences that do not satisfy monotonicity with respect to first-order stochastic dominance, thus separating the idea of probabilistically sophisticated choice from yet another tenet of subjective expected utility theory.

[3] A binary relation $\geqslant$ on a set A is acyclic if $a_1 \geqslant a_2 \geqslant, \ldots, \geqslant a_n$ implies that $\neg(a_n \geqslant a_1)$, for all $\{a_1 \ldots a_n\} \subseteq A$, $n \in \mathbb{N}$. In Machina and Schmeidler (1995) the transitivity of the indifference relation implies that replacement acyclicity is an implication of their replacement axiom.

then

$$f \asymp \sum_{i=1}^{n} \tau_i p_i,$$

where $\tau_1 = \alpha_{n-1} \cdot \alpha_{n-2} \cdot \ldots \cdot \alpha_1$, $\tau_i = \alpha_{n-1} \cdot \alpha_{n-2} \cdot \ldots \cdot (1-\alpha_{i-1})$, $\tau_n = (1-\alpha_{n-1})$ and $p_i = f(s_i)$, $i = 2, \ldots, n$.

Corresponding to every act there is a set of constant acts (i.e., elements of ΔX) that are non-comparable to it. These sets of constant acts are the manifestation of the decision maker's ambiguous beliefs. The next axiom requires that these ambiguous beliefs be consistent in the sense that one act is strictly preferred over another if and only if each constant acts that is non-comparable to the former is strictly preferred to the constant act that is non-comparable to the latter that is obtained by the same reduction process. This does not mean that the sets of constant acts corresponding to the two acts under consideration are necessarily disjoint. It only requires that for every possible belief the decision maker entertains, there is no reversal of order of the ranking of the two constant acts obtained by the reduction of the original acts invoking that belief. To state the axiom, I introduce the following additional notations. Let $\Delta^n := \{\alpha \in [0,1]^n \mid \Sigma_{i=1}^n \alpha_i = 1\}$. For each $h \in H$ and $\alpha \in \Delta^n$, define $h^\alpha = \Sigma_{i=1}^n \alpha_i h(s_i)$. Informally, h^α is the lottery in ΔX induced by applying the reduction of compound lotteries to the compound lottery whose first stage is the probability distribution α on S and the second stage consists of the state-contingent lottery payoffs of the act h.

(A.6) (**Constant-act comparability**) For all $f, g \in H$, $f \succ g$ if and only if $f^\alpha \succ g^\alpha$ for all $\alpha \in \Delta^n$ such that $f \asymp f^\alpha$ and $g \asymp g^\alpha$.

3. Representations

3.1. Multi-prior multi-utility representation

The model admits two sources of ambiguity, tastes and beliefs. The first result is a representation of a preference relation whose incompleteness is due to both sources of ambiguity. The representation involves a set $\mathcal{V}$ of utility functions on ΔX and a set Π of probability measures on S such that one act is strictly preferred over another if and only if the utility of each lottery induced by the reduction of the former act is larger than that of the corresponding lottery induced by the reduction of latter act according to the every probability measure in Π and every utility function in $\mathcal{V}$.

To state the main result I invoke the following definitions: A function V is *mixture continuous* if $V(\alpha p + (1-\alpha) q)$ is continuous in α, for all $p, q \in \Delta X$. It is *strictly monotonic* if $V(p) > V(q)$ whenever p dominates q according to first-order stochastic dominance.

Theorem 1. *Let $\succ$ be a binary relation on H then the following two conditions are equivalent:*

(i) $\succ$ is bounded on X and satisfies (A.1)–(A.6).

(ii) There exist a set, $\mathcal{V}$, of real-valued, mixture continuous, strictly monotonic functions, V on ΔX,and a convex set, Π, of probability measures on S such that, for all $f, g \in H$,

$$f \succ g \Leftrightarrow V(\Sigma_{s\in S}\pi(s) f(s)) > V(\Sigma_{s\in S}\pi(s) g(s)), \forall (V,\pi) \in \mathcal{V} \times \Pi. \tag{1}$$

and, for all $f \in H \backslash \{\delta^{\bar{x}}, \delta^{\underline{x}}\}$

$$V\left(\delta^{\bar{x}}\right) > V(\Sigma_{s\in S}\pi(s) f(s)) > V\left(\delta^{\underline{x}}\right), \forall (V,\pi) \in \mathcal{V} \times \Pi. \tag{2}$$

To describe the uniqueness of the representation (1) I adopt the notations of Evren and Ok (2011). Given any nonempty subset $\mathcal{U}$ of $\mathbb{R}^{\Delta X}$, define a map $\Gamma_{\mathcal{U}} : \Delta X \to \mathbb{R}^{\mathcal{U}}$ by $\Gamma_{\mathcal{U}}(p) = (u) := u(p)$. Note that for every $p \in \Delta X$, $u(p)$ is a real-valued function on $\mathcal{U}$.

Theorem 2. *Let $\succ$ be a binary relation on Hthat is bounded on X and satisfies (A.1)–(A.6). Two pairs of multi-utility multi-prior $(\mathcal{V}, \Pi)$ and $(\mathcal{V}^*, \Pi^*)$ represent $\succ$ if and only if $\Pi = \Pi^*$ and there exists $F : \Gamma_{\mathcal{V}}(\Delta X) \to \Gamma_{\mathcal{V}^*}(\Delta X)$ such that: (i) $\Gamma_{\mathcal{V}^*} = F \circ \Gamma_{\mathcal{V}}$ and (ii) for every $x, y \in \Gamma_{\mathcal{V}}(\Delta X)$, $x > y$ if and only if $F(x) > F(y)$.*

3.2. Complete tastes: Definition and representation

Consider the special case in which the decision maker is confident about her tastes and her indecisiveness is due solely to her ambiguous beliefs. This corresponds to the situation described by Bewley (2002) as Knightian uncertainty.[4] The next axiom rules out ambiguity regarding the decision maker's tastes.

(A.7) **(Complete tastes)** On the subset of constant acts in H, $\succ$ is negatively transitive.

With this in mind we have a probabilistically sophisticated version of Knightian uncertainty.

Theorem 3. *Let $\succ$ be a binary relation on H then the following two conditions are equivalent:*

(i) $\succ$ is bounded on X and satisfies (A.1)–(A.7).

(ii) There exist a real-valued, mixture continuous, strictly monotonic function V on ΔX and a convex set Π of probability measures on S such that, for all $f, g \in H$,

$$f \succ g \Leftrightarrow V(\Sigma_{s\in S}\pi(s) f(s)) > V(\Sigma_{s\in S}\pi(s) g(s)), \forall \pi \in \Pi \tag{3}$$

and, for all $f \in H \backslash \{\delta^{\bar{x}}, \delta^{\underline{x}}\}$,

$$V\left(\delta^{\bar{x}}\right) > V(\Sigma_{s\in S}\pi(s) f(s)) > V\left(\delta^{\underline{x}}\right), \forall \pi \in \Pi. \tag{4}$$

The function V is unique up to strictly monotonic increasing continuous transformation, and Π is unique.

3.3. Complete beliefs: Definition and representation

The next axiom, due originally to Galaabaatar and Karni (2013), formalizes the idea of complete beliefs. In other words, the decision maker's beliefs are characterized by a unique prior and her indecisiveness is due entirely to her ambiguous tastes.

(A.8) (**Complete beliefs**) For every event E and $\alpha \in [0,1]$, either $\alpha\delta^{\bar{x}} + (1-\alpha)\delta^{\underline{x}} \succ \delta_E^{\bar{x}}\delta^{\underline{x}}$ or $\delta_E^{\bar{x}}\delta^{\underline{x}} \succ \alpha'\delta^{\bar{x}} + (1-\alpha')\delta^{\underline{x}}$, for all $\alpha > \alpha'$.

The following theorem characterizes tastes ambiguity.

Theorem 4. *Let $\succ$ be a binary relation on H then the following two conditions are equivalent:*

(i) $\succ$ is bounded on X and satisfies (A.1)–(A.6) and (A.8).

(ii)There exist a set, $\mathcal{V}$ of real-valued, mixture continuous, strictly monotonic functions, Von ΔX and a probability measure π on$S$ such that, for all $f, g \in H$,

$$f \succ g \Leftrightarrow V(\Sigma_{s\in S}\pi(s) f(s)) > V(\Sigma_{s\in S}\pi(s) g(s)), \forall V \in \mathcal{V} \tag{5}$$

and, for all $f \in H \backslash \{\delta^{\bar{x}}, \delta^{\underline{x}}\}$,

$$V\left(\delta^{\bar{x}}\right) > V(\Sigma_{s\in S}\pi(s) f(s)) > V\left(\delta^{\underline{x}}\right), \forall V \in \mathcal{V}. \tag{6}$$

[4] Gilboa et al. (2010) depict the unanimity rule implied by Knightian uncertainty as a model of objective rationality.

Moreover, $\mathcal{V}^*$ is another set of utility functions on ΔX and a probability measure π^* on S that represent the preference relation $\succ$ in the sense of (5) if and only if $\pi = \pi^*$ and there exists $F : \Gamma_{\mathcal{V}}(\Delta X) \to \Gamma_{\mathcal{V}^*}(\Delta X)$ such that: (i) $\Gamma_{\mathcal{V}^*} = F \circ \Gamma_{\mathcal{V}}$ and (ii) for every $x, y \in \Gamma_{\mathcal{V}}(\Delta X)$, $x > y$ if and only if $F(x) > F(y)$.

4. Concluding remarks

4.1. Weak preferences and their representation

Given $\succ$ on H, Galaabaatar and Karni (2013) defined the weak preference relation $\succcurlyeq_{GK}$ and indifference relation $\sim_{GK}$ on H as follows: For all $f, g \in H$, $f \succcurlyeq_{GK} g$ if, for all $h \in H$, $h \succ f$ implies $h \succ g$, and $f \sim_{GK} g$ if $f \succcurlyeq_{GK} g$ and $g \succcurlyeq_{GK} f$. Note that $\succcurlyeq_{GK}$ on H is a preorder (that is, transitive and reflexive). According to these definitions, the representations of the weak preference and indifference relations that display both belief and tastes ambiguity, corresponding to (1) are as follows: For all f, $g \in H$,

$$f \succcurlyeq_{GK} g \Leftrightarrow V\left(\Sigma_{s\in S}\pi(s)f(s)\right) \geq V\left(\Sigma_{s\in S}\pi(s)g(s)\right), \forall (V,\pi) \in \mathcal{V}\times\Pi, \tag{7}$$

and

$$f \sim_{GK} g \Leftrightarrow V\left(\Sigma_{s\in S}\pi(s)f(s)\right) = V\left(\Sigma_{s\in S}\pi(s)g(s)\right), \forall (V,\pi) \in \mathcal{V}\times\Pi. \tag{8}$$

The corresponding representations of the weak preference and indifference relations in the special cases of belief ambiguity and ambiguity of tastes are obtained when the $\mathcal{V}$ and Π, respectively, are singleton sets.

4.2. A topological approach

In this paper I followed Machina and Schmeidler in employing the algebraic approach to modeling probabilistically sophisticated choice behavior. Alternatively, one could invoke a topological approach by imposing a topological structure on the choice space H and on the preference relation $\succcurlyeq_{GK}$ on H. Since the main point here is illustrative, to simplify the exposition suppose that X is finite and let ΔX be endowed with the $\mathbb{R}^n$ topology. Then, $H = (\Delta X)^n$ is a compact subset of a Euclidean space. Suppose that $\succcurlyeq_{GK}$ on H is a *continuous preorder* (that is, for all $f \in H$, the upper and lower contour sets $U_{\succcurlyeq_{GK}}(f) := \{h \in H \mid h \succcurlyeq_{GK} f\}$ and $L_{\succcurlyeq_{GK}}(f) := \{h \in H \mid f \succcurlyeq_{GK} h\}$, respectively, are closed and $\succcurlyeq_{GK}$ is a closed subset of $H \times H$).

The weak preference relation $\succcurlyeq_{GK}$ on H is said to have *continuous multi-utility representation* if it there is a set $\mathcal{U}$ of continuous real-valued functions on H such that $f \succcurlyeq_{GK} g$ if and only if $U(f) \geq U(g)$, for all $U \in \mathcal{U}$. Since, H is compact subset of Euclidean space and $\succcurlyeq_{GK}$ on H is a continuous preorder, it has continuous multi-utility representation (see Evren and Ok, 2011, Corollary 3). Applying the argument in the proof of Lemma 3, there exists a set $\Pi \subset \Delta S$ of additive probability measures on S such that, for all $f, g \in H$, $f \succcurlyeq_{GK} g$ if and only if $\Sigma_{i=1}^n \pi(s_i) f(s_i) \succcurlyeq_{GK} \Sigma_{i=1}^n \pi(s_i) g(s_i)$, for all $\pi \in \Pi$. Combining these results we obtain the following:

Corollary. *Let $\succcurlyeq_{GK}$ be a binary relation on H then $\succcurlyeq_{GK}$ is continuous preorder that is bounded on X and satisfies monotonicity, replacement, replacement acyclicity and constant act comparability if and only if there exist a convex set, $\mathcal{V}$, of real-valued, continuous, strictly monotonic functions, V on ΔX, and a convex set, Π, of probability measures on S such that, for all $f, g \in H$,*

$$f \succcurlyeq_{GK} g \Leftrightarrow V\left(\Sigma_{s\in S}\pi(s)f(s)\right) \geq V\left(\Sigma_{s\in S}\pi(s)g(s)\right), \forall (V,\pi) \in \mathcal{V}\times\Pi. \tag{9}$$

and, for all $f \in H\backslash\{\delta^{\bar{x}}, \delta^{\underline{x}}\}$,

$$V\left(\delta^{\bar{x}}\right) > V\left(\Sigma_{s\in S}\pi(s)f(s)\right) > V\left(\delta^{\underline{x}}\right), \forall (V,\pi) \in \mathcal{V}\times\Pi. \tag{10}$$

Finally we note that Evren and Ok (2011) show that, under appropriate topological conditions the representation (9) can be extended to more general choice sets.

5. Proofs

5.1. Proof of Theorem 1

(*i*) $\Rightarrow$ (*ii*). Sufficiency is an implication of the following lemmata:

Lemma 1. *For each $f \in H\backslash\{\delta^{\bar{x}}, \delta^{\underline{x}}\}$ the set $A(f) := \{\alpha \in [0,1] \mid f \asymp \alpha\delta^{\bar{x}} + (1-\alpha)\delta^{\underline{x}}\}$ is a closed interval, $\left[\underline{\alpha}_f, \bar{\alpha}_f\right] \subseteq [0,1]$.*

Proof. Let $\underline{\alpha}_f = \sup\{\alpha \in [0,1] \mid f \succ \alpha\delta^{\bar{x}} + (1-\alpha)\delta^{\underline{x}}\}$. That $\underline{\alpha}_f$ exists follows form the fact that the set is bounded and is non-empty ($\alpha = 0$ is in the set). Moreover, by (A.3), $\underline{\alpha}_f$ is unique.[5] By similar argument, $\bar{\alpha}_f := \inf\{\alpha \in [0,1] \mid \alpha\delta^{\bar{x}} + (1-\alpha)\delta^{\underline{x}} \succ f\}$ exists and is unique.

Next we show that $f \asymp \bar{\alpha}_f\delta^{\bar{x}} + \left(1-\bar{\alpha}_f\right)\delta^{\underline{x}}$ (that is, $\neg(f \succ \bar{\alpha}_f\delta^{\bar{x}} + \left(1-\bar{\alpha}_f\right)\delta^{\underline{x}})$ and $\neg(\bar{\alpha}_f\delta^{\bar{x}} + \left(1-\bar{\alpha}_f\right)\delta^{\underline{x}} \succ f)$). If $f \succ \bar{\alpha}_f\delta^{\bar{x}} + \left(1-\bar{\alpha}_f\right)\delta^{\underline{x}}$ then, since $\delta^{\bar{x}} \succ f$, by (A.2) there exists $\beta > \bar{\alpha}_f$ such that $f \succ \beta\delta^{\bar{x}} + (1-\beta)\delta^{\underline{x}}$. But, by (A.3) and the definition of $\bar{\alpha}_f$, $\beta > \bar{\alpha}_f$ implies that $\beta\delta^{\bar{x}} + (1-\beta)\delta^{\underline{x}} \succ f$. A contradiction. If $\bar{\alpha}_f\delta^{\bar{x}} + \left(1-\bar{\alpha}_f\right)\delta^{\underline{x}} \succ f$ then, since $f \succ \delta^{\underline{x}}$, by (A.2) there is $\beta < \bar{\alpha}_f$ such that $\beta\delta^{\bar{x}} + (1-\beta)\delta^{\underline{x}} \succ f$. This contradicts the definition of $\bar{\alpha}_f$. Hence, $f \asymp \bar{\alpha}_f\delta^{\bar{x}} + \left(1-\bar{\alpha}_f\right)\delta^{\underline{x}}$. By a similar argument, $f \asymp \underline{\alpha}_f\delta^{\bar{x}} + \left(1-\underline{\alpha}_f\right)\delta^{\underline{x}}$.

Let $\alpha \in \left(\underline{\alpha}_f, \bar{\alpha}_f\right)$ then, by definition of $\bar{\alpha}_f$ and $\underline{\alpha}_f$, respectively, $\neg(f \succ \alpha\delta^{\bar{x}} + (1-\alpha)\delta^{\underline{x}})$ and $\neg(\alpha\delta^{\bar{x}} + (1-\alpha)\delta^{\underline{x}} \succ f)$. Hence, $f \asymp \alpha\delta^{\bar{x}} + (1-\alpha)\delta^{\underline{x}}$. Combining these results we conclude that $f \asymp \alpha\delta^{\bar{x}} + (1-\alpha)\delta^{\underline{x}}$ for all $\alpha \in \left[\underline{\alpha}_f, \bar{\alpha}_f\right]$. ▲

Lemma 2. *There exists a convex set, $\mathcal{V}$, of strictly monotonic, mixture continuous functions $V : H \to [0,1]$ such that, for all $f, g \in H$, $f \succ g$ if and only if $V(f) > V(q)$, for all $V \in \mathcal{V}$.*

Proof. For each $v \in [0,1]$ define a function $V : H \to [0,1]$ by $V(f) = v\bar{\alpha}_f + (1-v)\underline{\alpha}_f$, for $f \in H$. Let $\mathcal{V} := \{V \mid v \in [0,1]\}$.

Suppose that $f \succ g$. Since $g \succ \delta^{\underline{x}}$, by (A.2) and (A.3), for every $\alpha \in (0,1)$ such that $\alpha\delta^{\bar{x}} + (1-\alpha)\delta^{\underline{x}} \succ f$ there is $\alpha' \in (0,\alpha)$ such that $\alpha\delta^{\bar{x}} + (1-\alpha)\delta^{\underline{x}} \succ \alpha'\delta^{\bar{x}} + \left(1-\alpha'\right)\delta^{\underline{x}} \succ g$. By definition of $\bar{\alpha}_f$ there is $\hat{\alpha} \leq \bar{\alpha}_f$ such that $\hat{\alpha}\delta^{\bar{x}} + \left(1-\hat{\alpha}\right)\delta^{\underline{x}} \succ g$. Thus, by definition of $\bar{\alpha}_g$, $\hat{\alpha}\delta^{\bar{x}} + \left(1-\hat{\alpha}\right)\delta^{\underline{x}} \succ \bar{\alpha}_g\delta^{\bar{x}} + \left(1-\bar{\alpha}_g\right)\delta^{\underline{x}}$. Hence, by (A.3), $\bar{\alpha}_f \geq \hat{\alpha} > \bar{\alpha}_g$. By similar argument, $\underline{\alpha}_f > \underline{\alpha}_g$. Consequently, by definition of $\mathcal{V}$, $V(f) > V(g)$, for all $V \in \mathcal{V}$.

Suppose that $V(f) > V(g)$ for all $V \in \mathcal{V}$. If $g \succ f$ then, by sufficiency, $V(g) > V(f)$, for all $V \in \mathcal{V}$, a contradiction. Thus, $\neg(g \succ f)$. If $g \asymp f$ then $V(f) > V(g)$ of some $V \in \mathcal{V}$ and $V(g) > V(f)$ of some $V \in \mathcal{V}$. In other words, $\left[\underline{\alpha}_f, \bar{\alpha}_f\right] \cap \left[\underline{\alpha}_g, \bar{\alpha}_g\right] \neq \varnothing$. Let

[5] To see this, suppose by way of negation that there are $\underline{\alpha}'_f > \underline{\alpha}_f$ that satisfy the definition. Let $\alpha' \in \left(\underline{\alpha}_f, \underline{\alpha}'_f\right)$ then, by (A.3), $\underline{\alpha}'_f\delta^{\bar{x}} + \left(1-\underline{\alpha}'_f\right)\delta^{\underline{x}} \succ \alpha'\delta^{\bar{x}} + \left(1-\alpha'\right)\delta^{\underline{x}} \succ \underline{\alpha}_f\delta^{\bar{x}} + \left(1-\underline{\alpha}_f\right)\delta^{\underline{x}}$. By definition of $\underline{\alpha}'_f$, $f \succ \alpha'\delta^{\bar{x}} + \left(1-\alpha'\right)\delta^{\underline{x}} \succ \underline{\alpha}_f\delta^{\bar{x}} + \left(1-\underline{\alpha}_f\right)\delta^{\underline{x}}$, which is a contradiction.

$\hat{\upsilon} \in [\underline{\alpha}_f, \bar{\alpha}_f] \cap [\underline{\alpha}_g, \bar{\alpha}_g]$, and define $\hat{V}(f) = \hat{\upsilon}\bar{\alpha}_f + (1-\hat{\upsilon})\underline{\alpha}_f$ and $\hat{V}(g) = \hat{\upsilon}\bar{\alpha}_g + (1-\hat{\upsilon})\underline{\alpha}_g$. Then $\hat{V} \in \mathcal{V}$ satisfies $\hat{V}(g) = \hat{V}(f)$, a contradiction. Thus, $\neg(g \asymp f)$. But $\neg(g \succ f)$ and $\neg(g \asymp f)$ imply $f \succ g$.

To show that all $V \in \mathcal{V}$ are monotonic, let $p, q \in \Delta X$ such that $p >_1 q$. By (A.3), $p \succ q$. We identify p with the constant act that pays off p in all $s \in S$. Hence, $V(p) > V(q)$, for all $V \in \mathcal{V}$.

To show that all $V \in \mathcal{V}$ are mixture continuous we observe that, by (A.2), for all $f, g \in H$ and $\beta \in (0,1)$, $\beta f + (1-\beta)g$ is continuous in β (that is, if a sequence (β_n) converges to β then $\lim_{n\to\infty} \beta_n f + (1-\beta_n)g = \beta f + (1-\beta)g$). Moreover, $\bar{\alpha}_{\beta f+(1-\beta)g}$ is continuous in β (that is, if a sequence (β_n) converges to β then $\lim_{n\to\infty} \bar{\alpha}_{\beta_n f+(1-\beta_n)g} = \bar{\alpha}_{\beta f+(1-\beta)g}$. By the same argument $\underline{\alpha}_{\beta f+(1-\beta)g}$ is continuous in β. Since, for all $V \in \mathcal{V}$, $V(\beta f + (1-\beta)g) = \upsilon\bar{\alpha}_{\beta f+(1-\beta)g} + (1-\upsilon)\underline{\alpha}_{\beta f+(1-\beta)g}$, V is mixture continuous. ▲

Lemma 3. *There exists a set $\Pi \subset \Delta S$ of additive probability measures on S such that, for all $f, g \in H$, $f \succ g$ if and only if $\Sigma_{i=1}^n \pi(s_i) f(s_i) \succ \Sigma_{i=1}^n \pi(s_i) g(s_i)$, for all $\pi \in \Pi$.*

Proof. For each $E \subseteq S$ let $\Pi(E) := \{\pi(E) \in [0,1] \mid \delta^{\bar{x}}_{\{E\}}\delta^{\underline{x}} \asymp \pi(E)\delta^{\bar{x}} + (1-\pi(E))\delta^{\underline{x}}\}$. By (A.2) and (A.3), for each event E, $\Pi(E)$ is a well-defined closed and bounded interval in $[0,1]$. Moreover, since for E is nonnull $\delta^{\bar{x}}_{\{E\}}\delta^{\underline{x}} \succ \delta^{\underline{x}}$ it follows that E is null, that is, $\neg(\delta^{\bar{x}}_{\{E\}}\delta^{\underline{x}} \succ \delta^{\underline{x}})$ if and only if $\pi(E)\delta^{\bar{x}} + (1-\pi(E))\delta^{\underline{x}} \asymp \delta^{\underline{x}}$ if and only if $\pi(E) = 0$. Thus, if E is null then $\Pi(E)$ is a singleton set whose element is $\{\pi(E) = 0\}$.

Let $f \in H$ be a non-constant act. By repeated applications of the replacement axiom, (A.4), we get:

$$f \asymp (\alpha_1 f(s_1) + (1-\alpha_1) f(s_2))_{\{s_1,s_2\}} f \asymp$$
$$\asymp \alpha_2(\alpha_1 f(s_1) + (1-\alpha_1) f(s_2)) + (1-\alpha_2) f(s_3)_{\{s_1,s_2,s_3\}} f \asymp$$
$$\alpha_3(\alpha_2(\alpha_1 f(s_1) + (1-\alpha_1) f(s_2)) + (1-\alpha_2) f(s_3)) + (1-\alpha_3) f(s_4)_{\{s_1,s_2,s_3,s_4\}} f \asymp \ldots \asymp \sum_{i=1}^n \tau_i f(s_i),$$

where $\tau_1 = \alpha_{n-1}\cdot\alpha_{n-2}\cdot\ldots\cdot\alpha_1$, $\tau_i = \alpha_{n-1}\cdot\alpha_{n-2}\cdot\ldots\cdot(1-\alpha_{i-1})$, $i = 2,\ldots,(n-1)$, and $\tau_n = (1-\alpha_{n-1})$. In general, $(\tau_i)_{i=1}^n$ is not unique. Let T denote the set of $\tau := (\tau_i)_{i=1}^n$ constructed in this manner and, for each $\tau \in T$, let $\tau_i(\tau)$ denote the ith coordinate of τ. Then, for all $\tau \in T$, $\Sigma_{i=1}^n \tau_i(\tau) = 1$.

By replacement acyclicity, (A.5), $f \asymp \Sigma_{i=1}^n \tau_i(\tau) f(s_i)$, for all $\tau \in T$. Moreover, s_i is null if and only if for each $\tau \in T$, the i–th coordinate of τ, $\tau_i = 0$. For every event $E \in 2^S$ let $\tau(E) := \Sigma_{s_i\in E}\tau_i(\tau)$ and $T(E) := \{\tau(E) \mid \tau \in T\}$. By the argument above, $\delta^{\bar{x}}_E\delta^{\underline{x}} \asymp \tau(E)\delta^{\bar{x}} + (1-\tau(E))\delta^{\underline{x}}$, for all $E \in 2^S$ and $\tau(E) \in T(E)$. Hence, by definition of $\Pi(E)$, $T(E) = \Pi(E)$, for all $E \in 2^S$. Thus, $\Pi = T$ is the set of probability distributions on S such that $f \asymp \Sigma_{i=1}^n \pi(s_i) f(s_i)$, for all $\pi \in \Pi$.

By (A.6), $f \asymp \Sigma_{i=1}^n \pi(s_i) f(s_i)$ and $g \asymp \Sigma_{i=1}^n \pi(s_i) g(s_i)$ for all $\pi \in \Pi$ imply that $f \succ g$ if and only if $\Sigma_{i=1}^n \pi(s_i) f(s_i) \succ \Sigma_{i-1}^n \pi(s_i) g(s_i)$ for all $\pi \in \Pi$. ▲

By Lemma 2, $\Sigma_{i=1}^n \pi(s_i) f(s_i) \succ \Sigma_{i=1}^n \pi(s_i) g(s_i)$ if and only if $V(\Sigma_{i=1}^n \pi(s_i) f(s_i)) > V(\Sigma_{i=1}^n \pi(s_i) g(s_i))$, for all $V \in \mathcal{V}$. Since, by Lemma 3, $f \succ g$ if and only if $\Sigma_{i=1}^n \pi(s_i) f(s_i) \succ \Sigma_{i=1}^n \pi(s_i)(s_i)$, for all $\pi \in \Pi$, it holds that $f \succ g$ if and only if $V(\Sigma_{i=1}^n \pi(s_i) f(s_i)) > V(\Sigma_{i=1}^n \pi(s_i) g(s_i))$, for all $(V, \pi) \in \mathcal{V}\times\Pi$. This proves the validity of (1).

To show that every $\pi \in \Pi$ is additive, consider an event E and the act $\delta^{\bar{x}}_E\delta^{\underline{x}}$. By the argument above, $\delta^{\bar{x}}_E\delta^{\underline{x}} \asymp \pi(E)\delta^{\bar{x}} + (1-\pi(E))\delta^{\underline{x}}$, for all $\pi \in \Pi$. But, by construction, $\delta^{\bar{x}}_E\delta^{\underline{x}} \asymp \Sigma_{s\in E}\pi(s)\delta^{\bar{x}} + \Sigma_{s\in S\setminus E}\pi(s)\delta^{\underline{x}}$, for all $\pi \in \Pi$. Hence, by (A.3), for all $E \subseteq S$, $\pi(E) = \Sigma_{s\in E}\pi(s)$, for all $\pi \in \Pi$. Thus, every $\pi \in \Pi$ is additive.

By definition of $\delta^{\bar{x}}$ and $\delta^{\underline{x}}$, $\bar{\alpha}_{\delta^{\bar{x}}} = \underline{\alpha}_{\delta^{\bar{x}}} = 1$ and $\bar{\alpha}_{\delta^{\underline{x}}} = \underline{\alpha}_{\delta^{\underline{x}}} = 0$. Hence, by (A.3), $V(\delta^{\bar{x}}) > V(\Sigma_{i=1}^n \pi(s_i) f(s_i)) > V(\delta^{\underline{x}})$ for all $V \in \mathcal{V}$ and $\pi \in \Pi$. This proves the validity of (2).

$(ii) \implies (i)$. That (A.1) and (A.2) hold is immediate and (A.3) is implied by the strict monotonicity of V. Given $f \in H$, $\neg(V(f) > V(\Sigma_{s\in S}\pi(s) f(s)))$ and $\neg((\Sigma_{s\in S}\pi(s) f(s)) > V(f))$, for all $V \in \mathcal{V}$ and $\pi \in \Pi$, if and only if $\neg(f \succ \Sigma_{s\in S}\pi(s) f(s))$ and $\neg(\Sigma_{s\in S}\pi(s) f(s) \succ f)$ for each $\pi \in \Pi$. Hence, $f \asymp \Sigma_{s\in S}\pi(s) f(s)$, for every $\pi \in \Pi$. That the replacement axiom, (A.4), holds is an immediate implication of the last observation and the additivity of π.

To show that (A.5) holds, let $f^0 = f$,

$$f^1 = (\alpha_1 f(s_1) + (1-\alpha_1) f(s_2))_{\{s_1,s_2\}} f,$$
$$f^2 = \alpha_2(\alpha_1 f(s_1) + (1-\alpha_1) f(s_2)) + (1-\alpha_2) f(s_3)_{\{s_1,s_2,s_3\}} f, \ldots, f^{n-1} = \sum_{i=1}^n \tau_i p_i.$$

Let $\mathcal{I}$ denote the set of closed intervals in $[0,1]$. Define a correspondence $\varphi : H \twoheadrightarrow \mathcal{I}$ by $\varphi(f) = \{\alpha \in [0,1] \mid f \asymp \alpha\delta^{\bar{x}} + (1-\alpha)\delta^{\underline{x}}\}$, $\forall f \in H$. By Lemma 1, $\varphi(f) = [\underline{\alpha}_f, \bar{\alpha}_f] \subseteq [0,1]$. But, for all $f, g \in H$, $f \asymp g$ if and only if $\varphi(f) \subseteq \varphi(g)$ or $\varphi(f) \supseteq \varphi(g)$. Consider the sequence of incomparable acts

$$f \asymp (\alpha_1 f(s_1) + (1-\alpha_1) f(s_2))_{\{s_1,s_2\}} f \asymp$$
$$\asymp \alpha_2(\alpha_1 f(s_1) + (1-\alpha_1) f(s_2)) + (1-\alpha_2) f(s_3)_{\{s_1,s_2,s_3\}} f \asymp$$
$$\alpha_3(\alpha_2(\alpha_1 f(s_1) + (1-\alpha_1) f(s_2)) + (1-\alpha_2) f(s_3)) + (1-\alpha_3) f(s_4)_{\{s_1,s_2,s_3,s_4\}} f \asymp \ldots \asymp \sum_{i=1}^n \tau_i f(s_i).$$

Since each two consecutive acts, f^k, f^{k+1} in this sequence are incomparable, either $\varphi(f^k) \subseteq \varphi(f^{k+1})$ or $\varphi(f^k) \supseteq \varphi(f^{k+1})$, $k = 0,\ldots,n-1$. Because none of these sets is empty, their intersection, $\cap_{k=1}^{n-1}\varphi(f^k)$, in nonempty. Consequently, there exist $V \in \mathcal{V}$ and $\pi \in \Pi$ such that

$$V(f) = V\left(\sum_{i=1}^n \pi_i p_i\right).$$

Hence, $f \asymp \sum_{i=1}^n \pi_i p_i$.

To show that (A.6) holds we note that by (1) $\Sigma_{s\in S}\pi(s) f(s) \succ \Sigma_{s\in S}\pi(s) g(s)$, for all $\pi \in \Pi$ if and only if $V(\Sigma_{s\in S}\pi(s) f(s)) > V(\Sigma_{s\in S}\pi(s) g(s))$, for all $(V, \pi) \in \mathcal{V}\times\Pi$. But, for all $f \in H$, $f \asymp f^\pi$ for all $\pi \in \Pi$. Hence, $V(\Sigma_{s\in S}\pi(s) f(s)) = V(f)$, for all $(V, \pi) \in \mathcal{V}\times\Pi$. Hence, $V(\Sigma_{s\in S}\pi(s) f(s)) > V(\Sigma_{s\in S}\pi(s) g(s))$, for all $(V, \pi) \in \mathcal{V}\times\Pi$, if and only if $V(f) > V(g)$ for all $V \in \mathcal{V}$. By Lemma 2, $V(f) > V(g)$ for all $V \in \mathcal{V}$ if and only if $f \succ g$. Thus, $f \succ g$ if and only if $\Sigma_{s\in S}\pi(s) f(s) \succ \Sigma_{s\in S}\pi(s) g(s)$, for all $\pi \in \Pi$. ■

5.2. Proof of Theorem 2

To prove the uniqueness, suppose that there exist another set, Π^*, of probability measures on S and a set, $\mathcal{V}^*$, of mixture continuous, strictly monotonic, utility functions that jointly represent the preference relation $\succ$, where Π^* is distinct from Π and $\mathcal{V}^*$ may or may not be distinct from $\mathcal{V}$. This supposition implies that there exist $\pi^* \in \Pi^*\setminus\Pi$ or $\pi \in \Pi\setminus\Pi^*$.

Since Π is a convex set, if $\pi^* \in \Pi^*\setminus\Pi$ then there exist $s \in S$ such that $\pi^*(s) > \pi(s)$, for all $\pi \in \Pi$. Hence, there is ρ such that $\pi^*(s) > \rho > \pi(s)$, for all $\pi \in \Pi$. Consider the act $\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}}$. By the argument in the proof of Lemma 2, $\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}} \asymp$

$\pi(s)\,\delta^{\bar{x}}+(1-\pi(s))\,\delta^{\underline{x}}$, for all $\pi\in\Pi$. By (A.3), $\rho\delta^{\bar{x}}+(1-\rho)\,\delta^{\underline{x}}\succ\pi(s)\,\delta^{\bar{x}}+(1-\pi(s))\,\delta^{\underline{x}}$, for all $\pi\in\Pi$. Hence, by (A.6), $\rho\delta^{\bar{x}}+(1-\rho)\,\delta^{\underline{x}}\succ\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}}$.

Since the lottery $\pi^*(s)\,\delta^{\bar{x}}+(1-\pi^*(s))\,\delta^{\underline{x}}$ strictly first-order stochastically dominates the lottery $\rho\delta^{\bar{x}}+(1-\rho)\,\delta^{\underline{x}}$, by strict monotonicity of V^*,

$$V^*\left(\pi^*(s)\,\delta^{\bar{x}}+\left(1-\pi^*(s)\right)\delta^{\underline{x}}\right)>V^*\left(\rho\delta^{\bar{x}}+(1-\rho)\,\delta^{\underline{x}}\right),\ \forall V^*\in\mathcal{V}^*. \tag{11}$$

But $\pi^*\in\Pi^*$, implies that $\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}}\asymp\pi^*(s)\,\delta^{\bar{x}}+(1-\pi^*(s))\,\delta^{\underline{x}}$. Hence, it is not true that $V^*(\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}})<V^*\left(\pi^*(s)\,\delta^{\bar{x}}+(1-\pi^*(s))\,\delta^{\underline{x}}\right)$, $\forall V^*\in\mathcal{V}^*$. Thus, for some $V^*\in\mathcal{V}^*$, $V^*(\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}})\geq V^*\left(\pi^*(s)\,\delta^{\bar{x}}+(1-\pi^*(s))\,\delta^{\underline{x}}\right)$. But $\rho\delta^{\bar{x}}+(1-\rho)\,\delta^{\underline{x}}\succ\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}}$ implies $V^*(\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}})<V^*\left(\rho\delta^{\bar{x}}+(1-\rho)\,\delta^{\underline{x}}\right)$, for all $V^*\in\mathcal{V}^*$. Hence, (11) implies that $V^*(\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}})<V^*\left(\pi^*(s)\,\delta^{\bar{x}}+(1-\pi^*(s))\,\delta^{\underline{x}}\right)$, $\forall V^*\in\mathcal{V}^*$. A contradiction.

The uniqueness of $\mathcal{V}$ is an implication of Evren and Ok (2011) Remark 1. ■

5.3. Proof of Theorem 3

$(i)\Rightarrow(ii)$. Define a binary relation $\succeq$ on ΔX by $p\succeq q$ if $\neg(q\succ p)$, for all $p,q\in\Delta X$. Axioms (A.1) and (A.6) imply that $\succeq$ is complete and transitive. Denote by $\sim$ the symmetric part of $\succeq$. For each $p\in\Delta X$ define v_p by $p\sim v_p\delta^{\bar{x}}+\left(1-v_p\right)\delta^{\underline{x}}$. Define a function $V:\Delta X\to[0,1]$ by $V(p)=v_p$. By (A.1)–(A.3) V is well-defined, mixture continuous, strictly monotonic function on ΔX, and $p\succeq q$ if and only if $V(p)\geq V(q)$.

By Theorem 1, there exists a unique set, Π, of probability measures on S such that, for all $f\in H$ and $\pi\in\Pi$, $f\asymp\Sigma_{i=1}^n\pi(s_i)f(s_i)$. For each $\pi\in\Pi$, define a function $V^\pi:H\to[0,1]$ by $V^\pi(f)=V\left(\Sigma_{i=1}^n\pi(s_i)f(s_i)\right)$. Let $\mathcal{V}:=\{V^\pi\mid\pi\in\Pi\}$. But, by Lemma 1, $f\succ g$ if and only if $V^\pi(f)>V^\pi(g)$, for all $V^\pi\in\mathcal{V}$. Hence, $f\succ g$ if and only if, $V(\Sigma_{i=1}^n\pi(s_i)f(s_i))\succ V(\Sigma_{i=1}^n\pi(s_i)g(s_i))$, for all $\pi\in\Pi$.

That $(ii)\Rightarrow(i)$ and the uniqueness of the representation follow from the corresponding parts in the proof of Theorem 1. ■

5.4. Proof of Theorem 4

$(i)\Rightarrow(ii)$. By (A.3), (A.8) and the argument in the proof of Lemma 2, Π is a singleton set. Thus, by Theorem 1, for all $f,g\in H$, $f\succ g\Leftrightarrow V(\Sigma_{s\in S}\pi(s)f(s))>V(\Sigma_{s\in S}\pi(s)g(s))$, $\forall V\in\mathcal{V}$, where $\mathcal{V}$ is a set of mixture continuous, strictly monotonic, real-valued functions on ΔX.

$(ii)\Rightarrow(i)$. That the axioms (A.1)–(A.6) are implied by the presentation follows from Theorem 1. To show that (A.8) holds, let $E\subseteq S$. By strict monotonicity and mixture continuity of V, there is a unique α^* such that $V\left(\alpha^*\delta^{\bar{x}}+(1-\alpha^*)\,\delta^{\underline{x}}\right)=V\left(\delta^{\bar{x}}_E\delta^{\underline{x}}\right)$, for all $V\in\mathcal{V}$. Thus, if $\alpha>\alpha^*$ then $V\left(\alpha\delta^{\bar{x}}+(1-\alpha)\,\delta^{\underline{x}}\right)>V\left(\delta^{\bar{x}}_E\delta^{\underline{x}}\right)$ and $V\left(\alpha\delta^{\bar{x}}+(1-\alpha)\,\delta^{\underline{x}}\right)\leq V\left(\delta^{\bar{x}}_E\delta^{\underline{x}}\right)$, otherwise. Hence, by (A.3) and strict monotonicity of V, $V\left(\alpha'\delta^{\bar{x}}+\left(1-\alpha'\right)\delta^{\underline{x}}\right)<V\left(\delta^{\bar{x}}_E\delta^{\underline{x}}\right)$ for all $\alpha^*>\alpha'$ and $V\in\mathcal{V}$. Thus, for each $E\subseteq S$, $\alpha\delta^{\bar{x}}+(1-\alpha)\,\delta^{\underline{x}}\succ\delta^{\bar{x}}_E\delta^{\underline{x}}$ or $\delta^{\bar{x}}_E\delta^{\underline{x}}\succ\alpha'\delta^{\bar{x}}+\left(1-\alpha'\right)\delta^{\underline{x}}$, for all $\alpha>\alpha'$.

The uniqueness follows from Theorem 1. ■

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