Probabilistic Sophistication without Completeness

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Abstract

This is a study of probabilistically sophisticated choice behavior with incomplete preferences. Invoking the analytical framework of Anscombe and Aumann (1963) and building on the work of Machina and Schmeidler (1995), the paper provides an axiomatic characterization of the general multi-prior multi-utility probabilistically sophisticated representation. In addition, the paper lays axiomatic foundations for two special cases: complete beliefs and complete tastes. In the former case, the incompleteness is due to ambiguous tastes and in latter case it is due to umbiguous beliefs.

Keywords: Incomplete preferences, Probabilistic sophistication, multi-prior multi-utility representation, ambiguous tastes, ambiguous beliefs.

JEL classification: D8, D81, D83

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1 Introduction

Choice-based definition of subjective probabilities presumes that when called upon to decide among courses of action whose consequences are not known in advance, decision makers form of beliefs about the likely realization of the consequences, and that these beliefs are quantifiable by probabilities. Because the beliefs are personal, their representation is dubbed subjective probabilities. Borel (1924), Ramsey (1931) and de Finetti (1937) were first to proposed the key idea that subjective probabilities may be inferred from the odds a decision maker is willing to offer when betting on events or the truth of propositions. This idea found its ultimate expression in the seminal works of Savage (1954) and Anscombe and Aumann (1963). A common feature of these works is that the subjective probabilities are defined in the context of expected utility theory. Consequently, these works confound the definition of subjective probabilities with the hypothesis that individual choice among uncertain prospects is representable by a functional that is linear in the probabilities. However, the representation of a decision maker's beliefs by subjective probabilities and the notion of expected utility maximizing choice behavior are two separate ideas.

Machina and Schmeidler (1992, 1995) severed this connection by proposing a model, dubbed probabilistic sophistication, in which choice-based subjective probabilities are defined without requiring that the decision maker's preferences respect the strictures of expected utility theory. According to Machina and Schmeidler subjective probabilities transform acts (that is, random variables on state space that take their values in the set of consequences) into lotteries (that is, the corresponding probability distributions on the set of consequences) and preferences are represented by a utility function over the set of lotteries.

An central tenet of both the expected utility models and the probabilistically sophisticated models is that all alternative courses of action are comparable. That this presumption is not tenable as a general depiction of real-life decision making was recognized by von Neumann and Morgenstern who wrote "It is conceivable – and may even in a way be more realistic – to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable." (von Neumann and Morgenstern [1947] p. 19). Aumann (1962) finds universal comparability not only inaccurate description of real-life decision making but also lacking normative appeal. In his words, "Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from a normative viewpoint." (Aumann [1962], p. 446). Empirically, the main manifestation of incomplete preferences is indecisiveness or inertia.

Considering the restrictive nature of completeness requirement, the objectives of this paper is to examine the implications of relaxing the completeness axiom for the existence and meaning of subjective probabilities in the probabilistic sophistication model of Machina and Schmeidler (1995) and to study the representations of ambiguous beliefs and tastes in this model. More specifically, invoking the analytical framework of Anscombe and Aumann (1963), I explore conditions under which incomplete preference relations admit a multi-utility multi-prior probabilistic sophisticated representation and explore the conditions that characterize two special cases: Knightian uncertainty and single-prior multi-utility representation. The former special case, first explored by Bewley (2002) in the context of expected utility theory, attributes the incompleteness to the decision maker's ambiguous beliefs and that latter, explored by Shapley and Baucells (1998) and Dubra, Maccheroni, and Ok (2004), to his ambiguous tastes.

The main analytical difficulty introduced by the incompleteness of the preference relation is due to the non-transitivity of the incomparability relation. When the preference relation is complete, indecisiveness arises only when the decision maker is indifferent among the alternatives under consideration. In Machina and Schmeidler (1995) the indifference is an equivalence relation, hence it is transitive. They exploit the transitivity of indifference to link general Anscombe-Aumann acts to constant acts that are convex combinations of the state-contingent payoffs of the original acts. This link is severed when the incomparability relation is non-transitive. To compensate for the loss of transitivity, the preference structure is enhanced by the introduction of two new axioms, dubbed replacement acyclicity and constant-act comparability. Replacement acyclicity requires that the incomparability relation restricted to replacement paths be acyclic. Constant-act comparability requires that one act be strictly preferred over another if and only if the induced constant acts that are non-comparable to the former are strictly preferred to those induced by the latter.

2 The Model

2.1 Analytical framework

Let $S = \{s_1..., s_n\}$ be a finite set of *states* and X a set of *outcomes*. Denote by ΔX the set of simple probability distributions on X. Elements of ΔX are *lotteries*. Let H be the set of mappings on S to ΔX. Elements of H are *acts* defined by Anscombe-Aumann (1963). The constant acts are identified with elements of ΔX, hence, $\Delta X \subset H$. Let $\succ$ be a binary relation on H referred to as *strict preference relation*. The relation $\succ$ is bounded on X if there are $\bar{x}$ and $\underline{x}$ in X such that $\delta^{\bar{x}} \succ \delta^{x} \succ \delta^{\underline{x}}$, for all $x \in X \backslash \{\bar{x}, \underline{x}\}$, where $\delta^x \in \Delta X$ is the degenerate lottery that assigns the unit probability mass to x. Define the *incomparability relation*, $\asymp$ on H as follows: For all $f, g \in H$, $f \asymp g$ if $\neg(f \succ g)$ and $\neg(g \succ f)$. Clearly, $\asymp$ is symmetric and reflexive but is not necessarily transitive.

2.2 The axiomatic structure

The following axioms depict the preference structure.

(A.1) (**Strict partial order**) The strict preference relation $\succ$ is transitive and irreflexive.

(A.2) (**Archimedean**) For all $f, g, h \in H$, if $f \succ g$ and $g \succ h$ then there exist $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1 - \alpha) h \succ g \succ \beta f + (1 - \beta) h$.

The statement of the next two axioms invokes the following additional notations and definitions. For every event E (that is, a subset of S) and $f, g \in H$, let $f_E g \in H$ be the a act that coincides with f on E and with g on $S \backslash E$. An event E is *nonnull* if $(\delta^{\bar{x}}_E f \succ \delta^{\underline{x}}_E f)$ and is *null* otherwise. Following Machina and Schmeidler (1995) the lottery p *is said to dominates the lottery* q *according to first-order stochastic dominance*, denoted $p >^1 q$, if $\Sigma_{\{x_s \succ x\}} p(x_s) \geq \Sigma_{\{x_s \succ x\}} q(x_s)$, for all $x \in X$ with strict inequality for some $x \in X$. The next axiom requires that first-order stochastically dominating lotteries be preferred.[1]

(A.3) (**Monotonicity**) For all $p, q \in \Delta X$, if $p >^1 q$ then $p_E h \succ q_E h$, for all nonnull $E \subset S$ and all $h \in H$.

[1] The same property, implied by their adoption of Savage's P3, is a tacit aspect of Machina and Schmeidler (1992). Grant (1995) characterized probabilistically sophisticated preferences that do not satisfies monotonicity with respect to first-order stochastic dominance, thus separating the idea of proabilistic sophisticated choice from yet another tenet of subjective expected utility theory.

The next axiom is a reformulation of the Horse/Roulette Replacement axiom of Machina and Schmeidler (1995) replacing the indifference with the incomparability relation.

(A.4) (**Replacement**) For every finite partition $(E_1, ..., E_n)$ of S, if

$$\delta^{\bar{x}}_{E_i}\left(\delta^{\underline{x}}_{E_j}\delta^{\underline{x}}\right) \asymp \left(\alpha\delta^{\bar{x}} + (1-\alpha)\,\delta^{\underline{x}}\right)_{E_i \cup E_j}\delta^{\underline{x}}$$

for some $\alpha \in [0,1]$ then

$$p_{E_i}\left(q_{E_j}h\right) \asymp (\alpha p + (1-\alpha)\,q)_{E_i \cup E_j}\,h$$

for all $p, q \in \Delta X$ and $h \in H$.

The following axiom imposes acyclicity of the incomparability relation along a replacement path.[2]

(A.5) (**Replacement acyclicity)** For all $f \in H$, there are $\alpha_1, \alpha_2, ..., \alpha_{n-1}$ such that if

$$f(s_1)_{\{s_1\}}\left(f(s_2)_{\{s_2\}}f\right) \asymp (\alpha_1 f(s_1) + (1-\alpha_1)\,f(s_2))_{\{s_1,s_2\}}\,f \asymp$$
$$\asymp \alpha_2\left(\alpha_1 f(s_1) + (1-\alpha_1)\,f(s_2)\right) + (1-\alpha_2)\,f(s_3)_{\{s_1,s_2,s_3\}}\,f \asymp$$
$$\alpha_3\left(\alpha_2\left(\alpha_1 f(s_1) + (1-\alpha_1)\,f(s_2)\right) + (1-\alpha_2)\,f(s_3)\right) + (1-\alpha_3)\,f(s_4)_{\{s_1,s_2,s_3,s_4\}}\,f \asymp ... \asymp \sum_{i=1}^{n}\tau_i p_i,$$

then

$$f \asymp \sum_{i=1}^{n}\tau_i p_i,$$

where $\tau_1 = \alpha_{n-1}\cdot\alpha_{n-2}\cdot...\cdot\alpha_1$, $\tau_i = \alpha_{n-1}\cdot\alpha_{n|-2}\cdot...\cdot(1-\alpha_{i-1})$, and $\tau_n = (1-\alpha_{n-1})$ and $p_1 = \alpha_1 f(s_1) + (1-\alpha_1)\,f(s_2)\,,\ ...,\ p_i = \alpha_i p_{i-1} + (1-\alpha_i)\,f(s_{i+1})\,,\ i = 2, ..., (n-1)\,.$

To state the next axiom, I introduce the following additional notations. Let $\Delta^n := \{\alpha \in [0,1]^n \mid \Sigma_{i=1}^n \alpha_i = 1\}$. For each $h \in H$ and $\alpha \in \Delta^n$, define $h^\alpha = \Sigma_{i=1}^n \alpha_i h(s_i)\,.$ Informally, h^α is the lottery in ΔX induced by applying the reduction of compound lotteries to compound lottery that whose first stage is the probability distribution α on S and the second stage consists of the state-contingent lottery payoffs of the act h.

(A.6) (**Constant-act comparability)** For all $f, g \in H$, $f \succ g$ if and only if $f^\alpha \succ g^\alpha$ for all $\alpha \in \Delta^n$ such that $f \asymp f^\alpha$ and $g \asymp g^\alpha$.

[2] A binary relation $\geqslant$ on a set A is acyclic if $a_1 \geqslant a_2 \geqslant, ..., \geqslant a_n$ implies that $a_1 \geqslant a_n$, for all $\{a_1...a_n\} \subseteq A.$ In Machina and Schmeidler (1995) the transitivity of the indifference relation implies that repacement acyclicity is an implication of their repalcement axiom.

3 Representations

3.1 Multi-prior multi-utility representation

The model admits two sources of ambiguity, namely, ambiguous tastes and ambiguous beliefs. The first result is a representation of a preference relation whose incompleteness is due to both sources of ambiguity. The representation involves a set $\mathcal{V}$ of utility functions on ΔX and a set Π of probability measures on S such that one act is strictly preferred over another if an only if the utility of each lottery induced by the reduction of the the former act is larger than that of the corresponding lotteries induced by the reduction of second act according to every the probability measure in Π.

To state the main result I invoke the following definitions: A function V is *mixture continuous* if $V(\alpha p + (1-\alpha) q)$ is continuous in α, for all $p, q \in \Delta X$. It is *strictly monotonic* if $V(p) \geq V(q)$ whenever p dominates q according to first-order stochastic dominance, with strict inequality in the case of strict dominance.

Theorem 1 *Let $\succ$ be a binary relation on H then the following two conditions are equivalent:*

(i) $\succ$ is bounded on X and satisfies (A.1) - (A.6).

(ii) There exist a set, $\mathcal{V}$, of real-valued, mixture continuous, strictly monotonic functions, V on ΔX, and a convex set, Π, of probability measures on S such that, for all $f, g \in H$,

$$f \succ g \Leftrightarrow V\left(\Sigma_{s\in S}\pi(s) f(s)\right) > V\left(\Sigma_{s\in S}\pi(s) g(s)\right), \forall (V, \pi) \in \mathcal{V} \times \Pi. \tag{1}$$

and, for all $f \in H$,

$$V\left(\delta^{\bar{x}}\right) > V\left(\Sigma_{s\in S}\pi(s) f(s)\right) > V\left(\delta^{\underline{x}}\right), \forall (V, \pi) \in \mathcal{V} \times \Pi. \tag{2}$$

To describe the uniqueness of the representation (1) I adopt the notations of Evren and Ok (2011). Given any nonempty subset $\mathcal{U}$ of $\mathbb{R}^{\Delta X}$, define a map $\Gamma_{\mathcal{U}} : \Delta X \to \mathbb{R}^{\mathcal{U}}$ by $\Gamma_{\mathcal{U}}(p) = (u) := u(p)$.

Theorem 2 *Let $\succ$ be a binary relation on H that is bounded on X and satisfies (A.1) - (A.6). Two pairs of multi-utility muti-prior $(\mathcal{V}, \Pi)$ and $(\mathcal{V}^*, \Pi^*)$ represent $\succ$ if and only if $\Pi = \Pi^*$ and there exists $F : \Gamma_{\mathcal{V}}(\Delta X) \to \Gamma_{\mathcal{V}^*}(\Delta X)$ such that: (i) $\Gamma_{\mathcal{V}^*} = F \circ \Gamma_{\mathcal{V}}$ and (ii) for every $x, y \in \Gamma_{\mathcal{V}}$, $x > y$ if and only if $F(x) > F(y)$.*

3.2 Complete tastes: Definition and representation

Consider the special case in which the decision maker is confident about her tastes and her indecisiveness is due solely to her ambiguous beliefs. This corresponds to the situation described by Bewley (2002) as Knightian uncertainty.[3] The next axiom rules out ambiguity regarding the decision maker's tastes.

(A.7) (Complete tastes) On the subset of constant acts in H, $\succ$ is negatively transitive.

With this in mind we have a probabilistically sophisticated version of Knightian uncertainty.

Theorem 3 *Let $\succ$ be a binary relation on H then the following two conditions are equivalent:*

(i) $\succ$ is bounded on X and satisfies (A.1) - (A.7).

(ii) There exist a real-valued, mixture continuous, strictly monotonic function V on ΔX and a convex set Π of probability measures on S such that, for all $f, g \in H$,

$$f \succ g \Leftrightarrow V\left(\Sigma_{s\in S}\pi\left(s\right)f\left(s\right)\right) \geq V\left(\Sigma_{s\in S}\pi\left(s\right)g\left(s\right)\right),\ \forall \pi \in \Pi \tag{3}$$

and, for all $f \in H$,

$$V\left(\delta^{\bar{x}}\right) > V\left(\Sigma_{s\in S}\pi\left(s\right)f\left(s\right)\right) > V\left(\delta^{\underline{x}}\right), \forall \pi \in \Pi. \tag{4}$$

The function V is unique up to strictly monotonic increasing continuous transformation, and Π is unique.

3.3 Complete beliefs: Definition and representation

The next axiom, due originally to Galaabaatar and Karni (2013), formalizes the idea of complete beliefs. In other words, the decision maker's beliefs are characterized by a unique prior and her indecisiveness is due entirely to her ambiguous tastes.

(A.8) (**Complete beliefs**) For all events E and $\alpha \in [0,1]$, either $\alpha\delta^{\bar{x}} + (1-\alpha)\,\delta^{\underline{x}} \succ \delta^{\bar{x}}_E\delta^{\underline{x}}$ or $\delta^{\bar{x}}_E\delta^{\underline{x}} \succ \alpha'\delta^{\bar{x}} + (1-\alpha')\,\delta^{\underline{x}}$, for all $\alpha > \alpha'$.

[3] See also Galaabaatar and Karni (2013). Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) depict the unanimity rule implied by Knightian uncertainty as a model of objective rationality.

The following theorem characterizes tastes ambiguity.

Theorem 4 *Let $\succ$ be a binary relation on H then the following two conditions are equivalent:*

(i) $\succ$ is bounded on X and satisfies (A.1) - (A.6) and (A.8).

(ii) There exist a set, $\mathcal{V}$, of real-valued, mixture continuous, strictly monotonic functions, V on ΔX and a probability measure π on S such that, for all $f, g \in H$,

$$f \succ g \Leftrightarrow V\left(\Sigma_{s\in S}\pi\left(s\right)f\left(s\right)\right) > V\left(\Sigma_{s\in S}\pi\left(s\right)g\left(s\right)\right),\ \forall V \in \mathcal{V} \tag{5}$$

and, for all $f \in H$,

$$V\left(\delta^{\bar{x}}\right) > V\left(\Sigma_{s\in S}\pi\left(s\right)f\left(s\right)\right) > V\left(\delta^{\underline{x}}\right), \forall V \in \mathcal{V}. \tag{6}$$

Moreover, $\mathcal{V}^*$ is another set of utility functions on ΔX and a probability measure π^* on S represent the preference relation $\succ$ in the sense of (5) if and only if $\pi = \pi^*$ and there exists $F : \Gamma_{\mathcal{V}}\left(\Delta X\right) \to \Gamma_{\mathcal{V}^*}\left(\Delta X\right)$ such that: (i) $\Gamma_{\mathcal{V}^*} = F \circ \Gamma_{\mathcal{V}}$ and (ii) for every $x, y \in \Gamma_{\mathcal{V}}$, $x > y$ if and only if $F\left(x\right) > F\left(y\right)$.

4 Concluding Remarks

4.1 Weak preferences and their representation

Given $\succ$ on H, Karni and Galaabaatar (2013) defined the weak preference relation $\succcurlyeq_{GK}$ and indifference relation $\sim_{GK}$ on H as follows: For all $f, g \in H$, $f \succcurlyeq_{GK} g$ if, for all $h \in H$, $h \succ f$ implies $h \succ g$, and $f \sim_{GK} g$ if $f \succcurlyeq_{GK} g$ and $g \succcurlyeq_{GK} f$. Note that $\succcurlyeq_{GK}$ on H is a preorder (that is, transitive and reflexive). According to these definitions, the representations of the weak preference and indifference relations that display both belief and tastes ambiguity, corresponding to (1) are as follows: For all $f, g \in H$,

$$f \succcurlyeq_{GK} g \Leftrightarrow V\left(\Sigma_{s\in S}\pi\left(s\right)f\left(s\right)\right) \geq V\left(\Sigma_{s\in S}\pi\left(s\right)g\left(s\right)\right), \forall\left(V,\pi\right) \in \mathcal{V}\times\Pi, \tag{7}$$

and

$$f \sim_{GK} g \Leftrightarrow V\left(\Sigma_{s\in S}\pi\left(s\right)f\left(s\right)\right) = V\left(\Sigma_{s\in S}\pi\left(s\right)g\left(s\right)\right), \forall\left(V,\pi\right) \in \mathcal{V}\times\Pi. \tag{8}$$

The corresponding representations of the weak preference and indifference relations in the special cases of belief ambiguity and ambiguity of tastes are obtained when the $\mathcal{V}$ and Π, respectively, are singleton sets.

4.2 A topological approach

In this paper I followed Machina and Schmeidler in employing the algebraic approach to modeling probabilistic sophisticated choice behavior. Alternatively, one could invoke a topological approach by imposing a topological structure on the choice space H and on the preference relation $\succcurlyeq_{GK}$ on H. Since the main point here is illustrative, to simplify the exposition suppose that X is finite and let ΔX be endowed with the $\mathbb{R}^n$ topology. Then, $H = (\Delta X)^n$ is a compact subset of a Euclidean space. Suppose that $\succcurlyeq_{GK}$ on H is a *continuous preorder* (that is, for all $f \in H$, the upper and lower contour sets $U_{\succcurlyeq_{GK}}(f) := \{h \in H \mid h \succcurlyeq_{GK} f\}$ and $L_{\succcurlyeq_{GK}}(f) := \{h \in H \mid f \succcurlyeq_{GK} h\}$, respectively, are closed and $\succcurlyeq_{GK}$ is a closed subset of $H \times H$).

The weak preference relation $\succcurlyeq_{GK}$ on H is said to have *continuous multi-utility representation* if it there is a set $\mathcal{U}$ of continuous real-valued functions on H such that $f \succcurlyeq_{GK} g$ if and only if $U(f) \geq U(g)$, for all $f, g \in H$. Since, H is compact subset of Euclidean space and $\succcurlyeq_{GK}$ on H is a continuous preorder, it has continuous multi-utility representation (see Evren and Ok [2011] Corollary 3). Applying the argument in the proof of Lemma 3, there exists a set $\Pi \subset \Delta S$ of additive probability measures on S such that, for all $f, g \in H$, $f \succcurlyeq_{GK} g$ if and only if $\Sigma_{i=1}^{n} \pi(s_i) f(s_i) \geq \Sigma_{i=1}^{n} \pi(s_i) g(s_i)$, for all $\pi \in \Pi$. Combining these results we obtain the following:

Corollary: *Let $\succcurlyeq_{GK}$ be a binary relation on H then $\succcurlyeq_{GK}$ is continuous preorder that is bounded on X and satisfies monotonicity, replacement, replacement acyclicity and constant act comparability if and only if there exist a convex set, $\mathcal{V}$, of real-valued, continuous, strictly monotonic functions, V on ΔX, and a convex set,* Π, *of probability measures on S such that, for all $f, g \in H$,*

$$f \succcurlyeq_{GK} g \Leftrightarrow V(\Sigma_{s\in S}\pi(s) f(s)) > V(\Sigma_{s\in S}\pi(s) g(s)), \forall (V, \pi) \in \mathcal{V} \times \Pi. \tag{9}$$

and, for all $f \in H$,

$$V\left(\delta^{\bar{x}}\right) > V(\Sigma_{s\in S}\pi(s) f(s)) > V(\delta^{\underline{x}}), \forall (V, \pi) \in \mathcal{V} \times \Pi. \tag{10}$$

Finally we note that Evren and Ok (2011) show that, under appropriate topological conditions the representation (9) can be extended to more general choice sets.

5 Proofs

5.1 Proof of Theorem 1

$(i) \Rightarrow (ii)$. Sufficiency is an implication of the following lemmata:

Lemma 1: For each $f \in H$ the set $A(f) := \{\alpha \in [0,1] \mid f \asymp \alpha\delta^{\bar{x}} + (1-\alpha)\,\delta^{\underline{x}}\}$ is a closed interval, $[\underline{\alpha}_f, \bar{\alpha}_f] \subseteq [0,1]$.

Proof. Let $\underline{\alpha}_f = \sup\{\alpha \in [0,1] \mid f \succ \alpha\delta^{\bar{x}} + (1-\alpha)\,\delta^{\underline{x}}\}$. That $\underline{\alpha}_f$ exists follows form the fact that the set is bounded and is non-empty ($\alpha = 0$ is in the set). Moreover, by (A.3), $\underline{\alpha}_f$ is unique. By similar argument, $\bar{\alpha}_f := \inf\{\alpha \in [0,1] \mid \alpha\delta^{\bar{x}} + (1-\alpha)\,\delta^{\underline{x}} \succ f\}$ exists and is unique.

Next we show that $f \asymp \bar{\alpha}_f\delta^{\bar{x}} + (1-\bar{\alpha}_f)\,\delta^{\underline{x}}$ (that is, $\neg(f \succ \bar{\alpha}_f\delta^{\bar{x}} + (1-\bar{\alpha}_f)\,\delta^{\underline{x}})$ and $\neg(\bar{\alpha}_f\delta^{\bar{x}} + (1-\bar{\alpha}_f)\,\delta^{\underline{x}} \succ f)$). If $f \succ \bar{\alpha}_f\delta^{\bar{x}} + (1-\bar{\alpha}_f)\,\delta^{\underline{x}}$ then, since $\delta^{\bar{x}} \succ f$, by (A.2) there exist $\beta > \bar{\alpha}_f$ such that $f \succ \beta\delta^{\bar{x}} + (1-\beta)\,\delta^{\underline{x}}$. But, by (A.3) and the definition of $\bar{\alpha}_f$, $\beta > \bar{\alpha}_f$ implies that $\beta\delta^{\bar{x}} + (1-\beta)\,\delta^{\underline{x}} \succ f$. A contradiction. If $\bar{\alpha}_f\delta^{\bar{x}} + (1-\bar{\alpha}_f)\,\delta^{\underline{x}} \succ f$ then, since $f \succ \delta^{\underline{x}}$, by (A.2) there is $\beta < \bar{\alpha}_f$ such that $\beta\delta^{\bar{x}} + (1-\beta)\,\delta^{\underline{x}} \succ f$. This contradicts the definition of $\bar{\alpha}_f$. Hence, $f \asymp \bar{\alpha}_f\delta^{\bar{x}} + (1-\bar{\alpha}_f)\,\delta^{\underline{x}}$. By a similar argument, $f \asymp \underline{\alpha}_f\delta^{\bar{x}} + \left(1-\underline{\alpha}_f\right)\delta^{\underline{x}}$.

Let $\alpha \in \left(\underline{\alpha}_f, \bar{\alpha}_f\right)$ then, by definition of $\bar{\alpha}_f$ and $\underline{\alpha}_f$, respectively, $\neg(f \succ \alpha\delta^{\bar{x}} + (1-\alpha)\,\delta^{\underline{x}})$ and $\neg(\alpha\delta^{\bar{x}} + (1-\alpha)\,\delta^{\underline{x}} \succ f)$. Hence, $f \asymp \alpha\delta^{\bar{x}} + (1-\alpha)\,\delta^{\underline{x}}$. Combining these results we conclude that $f \asymp \alpha\delta^{\bar{x}} + (1-\alpha)\,\delta^{\underline{x}}$ for all $\alpha \in \left[\underline{\alpha}_f, \bar{\alpha}_f\right]$. $\triangle$

Lemma 2: There exists a convex set, $\mathcal{V}$, of strictly monotonic, mixture continuous functions $V : H \to [0,1]$ such that, for all $f, g \in H$, $f \succ g$ if and only if $V(f) > V(q)$, for all $V \in \mathcal{V}$.

Proof. For each $v \in [0,1]$ define a function $V : H \to [0,1]$ by $V(f) = v\bar{\alpha}_f + (1-v)\,\underline{\alpha}_f$, for $f \in H$. Let $\mathcal{V} := \{V \mid v \in [0,1]\}$.

Suppose that $f \succ g$. Since $g \succ \delta^{\underline{x}}$, by (A.2) and (A.3), for every $\alpha \in (0,1)$ such that $\alpha\delta^{\bar{x}} + (1-\alpha)\,\delta^{\underline{x}} \succ f$ there is $\alpha' \in (0,\alpha)$ such that $\alpha\delta^{\bar{x}} + (1-\alpha)\,\delta^{\underline{x}} \succ \alpha'\delta^{\bar{x}} + (1-\alpha')\,\delta^{\underline{x}} \succ g$. By definition of $\bar{\alpha}_f$ there is $\hat{\alpha} \leq \bar{\alpha}_f$ such that $\hat{\alpha}\delta^{\bar{x}} + (1-\hat{\alpha})\,\delta^{\underline{x}} \succ g$. Thus, by definition of $\bar{\alpha}_g$, $\hat{\alpha}\delta^{\bar{x}} + (1-\hat{\alpha})\,\delta^{\underline{x}} \succ \bar{\alpha}_g\delta^{\bar{x}} + (1-)\,\delta\bar{\alpha}_g^{\underline{x}}$. Hence, by (A.3), $\bar{\alpha}_f \geq \hat{\alpha} > \bar{\alpha}_g$. By similar argument, $\underline{\alpha}_f > \underline{\alpha}_g$. Consequently, by definition of $\mathcal{V}$, $V(f) > V(g)$, for all $V \in \mathcal{V}$.

Suppose that $V(f) > V(g)$ for all $V \in \mathcal{V}$. If $g \succ f$ then, by sufficiency, $V(g) > V(f)$, for all $V \in \mathcal{V}$, a contradiction. Thus, $\neg(g \succ f)$. If $g \asymp f$ then there is $\hat{V} \in \mathcal{V}$ such that $\hat{V}(g) = \hat{V}(f)$, a contradiction. Thus, $\neg(g \asymp f)$. But $\neg(g \succ f)$ and $\neg(g \asymp f)$ imply $f \succ g$.

To show that all $V \in \mathcal{V}$ are monotonic, let $p, q \in \Delta X$ such that $p >_1 q$. By (A.3), $p \succ q$.

We identify p with the constant act that pays off p in all $s \in S$. Hence, $V(p) > V(q)$,for all $V \in \mathcal{V}$.

To show that all $V \in \mathcal{V}$ are mixture continuous we observe that, by (A.2), for all $f, g \in H$ and $\beta \in (0,1)$, $\beta f + (1-\beta) g$ is continuous in β (that is, if a sequence (β_n) converges to β then $\lim_{n\to\infty} \beta_n f + (1-\beta_n) g = \beta f + (1-\beta) g$). Moreover, $\bar{\alpha}_{\beta f+(1-\beta)g}$ is continuous in β (that is, if a sequence (β_n) converges to β then $\lim_{n\to\infty} \bar{\alpha}_{\beta_n f+(1-\beta_n)g} = \bar{\alpha}_{\beta f+(1-\beta)g}$. By the same argument $\underline{\alpha}_{\beta f+(1-\beta)g}$ is continuous in β. Since, for all $V \in \mathcal{V}$, $V(\beta f + (1-\beta) g) = v\bar{\alpha}_{\beta f+(1-\beta)g} + (1-v)\underline{\alpha}_{\beta f+(1-\beta)g}$, V is mixture continuous. ▲

Lemma 3. There exists a set $\Pi \subset \Delta S$ of additive probability measures on S such that, for all $f, g \in H$, $f \succ g$ if and only if $\Sigma_{i=1}^n \pi(s_i) f(s_i) \succ \Sigma_{i=1}^n \pi(s_i) g(s_i)$, for all $\pi \in \Pi$.

Proof. For each $E \subseteq S$ let $\Pi(E) := \{\pi(E) \in [0,1] \mid \delta^{\bar{x}}_{\{E\}} \delta^{\underline{x}} \asymp \pi(E)\delta^{\bar{x}} + (1-\pi(E))\delta^{\underline{x}}\}$. By (A.2) and (A.3), for each event E, $\Pi(E)$ is a well-defined closed and bounded interval in $[0,1]$. Moreover, since for E is nonnull $\delta^{\bar{x}}_{\{E\}}\delta^{\underline{x}} \succ \delta^{\underline{x}}$, it follows that E is null (that is, $\neg\left(\delta^{\bar{x}}_{\{E\}}\delta^{\underline{x}} \succ \delta^{\underline{x}}\right)$) if and only if $\pi(E) = 0$, for all $\pi \in \Pi(E)$.

Let $f \in H$ be a non-constant act. By repeated applications of the replacement axiom, (A.4), we get:

$$f \asymp (\alpha_1 f(s_1) + (1-\alpha_1) f(s_2))_{\{s_1,s_2\}} f \asymp$$

$$\asymp \alpha_2 (\alpha_1 f(s_1) + (1-\alpha_1) f(s_2)) + (1-\alpha_2) f(s_3)_{\{s_1,s_2,s_3\}} f \asymp$$

$$\alpha_3 (\alpha_2 (\alpha_1 f(s_1) + (1-\alpha_1) f(s_2)) + (1-\alpha_2) f(s_3)) + (1-\alpha_3) f(s_4)_{\{s_1,s_2,s_3,s_4\}} f \asymp ... \asymp \sum_{i=1}^n \tau_i f(s_i),$$

where $\tau_1 = \alpha_{n-1} \cdot \alpha_{n-2} \cdot ... \cdot \alpha_1$, $\tau_i = \alpha_{n-1} \cdot \alpha_{n-2} \cdot ... \cdot (1-\alpha_{i-1})$, $i = 2, ..., (n-1)$, and $\tau_n = (1-\alpha_{n-1})$. In general, $(\tau_i)_{i=1}^n$ is not unique. Let T denote the set of $\tau := (\tau_i)_{i=1}^n$ constructed in this manner then, for all $\tau \in T$, $\Sigma_{i=1}^n \tau_i = 1$.

By replacement acyclicity, (A.5), $f \asymp \Sigma_{i=1}^n \tau_i f(s_i)$, for all $\tau \in T$. Moreover, s_i is null if and only if for each $\tau \in T$, the $i-$th coordinate of τ, $\tau_i = 0$. For each $s_j \in S$ let $T(s_j) := \{\tau_j \mid (\tau_1, ..., \tau_j, ..., \tau_n) \in T\}$. Consider the act $\delta^{\bar{x}}_{\{s_i\}}\delta^{\underline{x}}$. By the argument above, $\delta^{\bar{x}}_{\{s_i\}}\delta^{\underline{x}} \asymp \tau_i \delta^{\bar{x}} + (1-\tau_i)\delta^{\underline{x}}$, for $i = 1, ..., n$ and $\tau_i \in T(s_i)$. Hence, by definition, $T(s_i) = \Pi(\{s_i\})$, for all $i = 1, ..., n$, and $\Pi = T$. Then Π is the set of probability distributions on S such that $f \asymp \Sigma_{i=1}^n \pi(s_i) f(s_i)$, for all $\pi \in \Pi$.

By (A.6), $f \asymp \Sigma_{i=1}^n \pi(s_i) f(s_i)$ and $g \asymp \Sigma_{i=1}^n \pi(s_i) g(s_i)$ for all $\pi \in \Pi$ imply that $f \succ g$ if and only if $\Sigma_{i=1}^n \pi(s_i) f(s_i) \succ \Sigma_{i-1}^n \pi(s_i) g(s_i)$ for all $\pi \in \Pi$. ▲

By Lemma 2, $\Sigma_{i=1}^{n|}\pi(s_i) f(s_i) \succ \Sigma_{i=1}^{n}\pi(s_i) g(s_i)$ if and only if $V(\Sigma_{i=1}^{n}\pi(s_i) f(s_i)) > V(\Sigma_{i=1}^{n}\pi(s_i) g(s_i))$, for all $V \in \mathcal{V}$. Since, by Lemma 3, $f \succ g$ if and only if $\Sigma_{i=1}^{n}\pi(s_i) f(s_i) \succ \Sigma_{i=1}^{n}\pi(s_i)(s_i)$, for all $\pi \in \Pi$, it holds that $f \succ g$ if and only if $V(\Sigma_{i=1}^{n}\pi(s_i) f(s_i)) > V(\Sigma_{i=1}^{n}\pi(s_i) g(s_i))$, for all $(V,\pi) \in \mathcal{V}\times\Pi$. The proves the validity of (1).

To show that every $\pi \in \Pi$ is additive, consider an event E and the act $\delta_E^{\bar{x}}\delta^{\underline{x}}$. By the argument above, $\delta_E^{\bar{x}}\delta^{\underline{x}} \asymp \pi(E)\delta^{\bar{x}} + (1-\pi(E))\delta^{\underline{x}}$, for all $\pi \in \Pi$. But, by construction, $\delta_E^{\bar{x}}\delta^{\underline{x}} \asymp \Sigma_{s\in E}\pi(s)\delta^{\bar{x}} + \Sigma_{s\in S\setminus E}\pi(s)\delta^{\underline{x}}$, for all $\pi \in \Pi$. Hence, by (A.3), for all $E \subseteq S$, $\pi(E) = \Sigma_{s\in E}\pi(s)$, for all $\pi \in \Pi$. Thus, every $\pi \in \Pi$ is additive.

By definition of $\delta^{\bar{x}}$ and $\delta^{\underline{x}}$, $\bar{\alpha}_{\delta^{\bar{x}}} = \underline{\alpha}_{\delta^{\bar{x}}} = 1$ and $\bar{\alpha}_{\delta^{\underline{x}}} = \underline{\alpha}_{\delta^{\underline{x}}} = 0$. Hence, by (A.3), $V\left(\delta^{\bar{x}}\right) > V\left(\Sigma_{i=1}^{n}\pi(s_i) f(s_i)\right) > V\left(\delta^{\underline{x}}\right)$ for all $V \in \mathcal{V}$ and $\pi \in \Pi$. This proves the validity of (2).

$(ii) \Longrightarrow (i)$. That (A.1) and (A.2) hold is immediate and (A.3) is implied by the strict monotonicity of V.

Given $f \in H$, $\neg(V(f) > V(\Sigma_{s\in S}\pi(s) f(s)))$ and $\neg((\Sigma_{s\in S}\pi(s) f(s)) > V(f))$, for all $V \in \mathcal{V}$ and $\pi \in \Pi$, if and only if $\neg(f \succ \Sigma_{s\in S}\pi(s) f(s))$ and $\neg(\Sigma_{s\in S}\pi(s) f(s) \succ f)$ for each $\pi \in \Pi$. Hence, $f \asymp \Sigma_{s\in S}\pi(s) f(s)$, for every $\pi \in \Pi$. That the replacement axiom, (A.4), holds is an immediate implication of the last observation and the additivity of π.

To show that (A.5) holds, let $f^0 = f$,

$$f^1 = (\alpha_1 f(s_1) + (1-\alpha_1) f(s_2))_{\{s_1,s_2\}} f,$$

$$f^2 = \alpha_2 (\alpha_1 f(s_1) + (1-\alpha_1) f(s_2)) + (1-\alpha_2) f(s_3)_{\{s_1,s_2,s_3\}} f, ..., f^{n-1} = \sum_{i=1}^{n} \tau_i p_i.$$

Let $\mathcal{I}$ denotes the set of closed intervals in $[0,1]$. Define a correspondence $\varphi : H \twoheadrightarrow \mathcal{I}$ by $\varphi(f) = \{\alpha \in [0,1] \mid f \asymp \alpha\delta^{\bar{x}} + (1-\alpha)\delta^{\underline{x}}\}$, $\forall f \in H$. By Lemma 1, $\varphi(f) = \left[\underline{\alpha}_f, \bar{\alpha}_f\right] \subseteq [0,1]$. But, for all $f, g \in H$, $f \asymp g$ if and only if $\varphi(f) \subseteq \varphi(g)$ or $\varphi(f) \supseteq \varphi(g)$. Consider the sequence of incomparable acts

$$f \asymp (\alpha_1 f(s_1) + (1-\alpha_1) f(s_2))_{\{s_1,s_2\}} f \asymp$$

$$\asymp \alpha_2 (\alpha_1 f(s_1) + (1-\alpha_1) f(s_2)) + (1-\alpha_2) f(s_3)_{\{s_1,s_2,s_3\}} f \asymp$$

$$\alpha_3 (\alpha_2 (\alpha_1 f(s_1) + (1-\alpha_1) f(s_2)) + (1-\alpha_2) f(s_3)) + (1-\alpha_3) f(s_4)_{\{s_1,s_2,s_3,s_4\}} f \asymp ... \asymp \sum_{i=1}^{n} \tau_i f(s_i).$$

Since each two consecutive acts, f^k, f^{k+1} in this sequence are incomparable, either $\varphi\left(f^k\right) \subseteq \varphi\left(f^{k+1}\right)$ or $\varphi\left(f^k\right) \supseteq \varphi\left(f^{k+1}\right)$, $k = 0, ..., n-1$. Because none of these sets is empty, their intersection, $\cap_{k=1}^{n-1}\varphi\left(f^k\right)$, in nonempty. Consequently, there exist $V \in \mathcal{V}$ and $\pi \in \Pi$ such that

$$V(f) = V\left(\sum_{i=1}^{n} \pi_i p_i\right).$$

Hence, $f \asymp \sum_{i=1}^{n} \pi_i p_i$.

To show that (A.6) holds we note that by (1) $\Sigma_{s\in S}\pi(s) f(s) \succ \Sigma_{s\in S}\pi(s) g(s)$, for all $\pi \in \Pi$ if and only if $V(\Sigma_{s\in S}\pi(s) f(s)) > V(\Sigma_{s\in S}\pi(s) g(s))$, for all $(V, \pi) \in \mathcal{V} \times \Pi$. But, for all $f \in H$, $f \asymp f^\pi$ for all $\pi \in \Pi$. Hence, $V(\Sigma_{s\in S}\pi(s) f(s)) = V(f)$, for all $(V, \pi) \in \mathcal{V} \times \Pi$. Hence, $V(\Sigma_{s\in S}\pi(s) f(s)) > V(\Sigma_{s\in S}\pi(s) g(s))$, for all $(V, \pi) \in \mathcal{V} \times \Pi$, if and only if $V(f) > V(g)$ for all $V \in \mathcal{V}$. By Lemma 2, $V(f) > V(g)$ for all $V \in \mathcal{V}$ if and only if $f \succ g$. Thus, $f \succ g$ if and only if $\Sigma_{s\in S}\pi(s) f(s) \succ \Sigma_{s\in S}\pi(s) g(s)$, for all $\pi \in \Pi$. ■

5.2 Proof of Theorem 2

To prove the uniqueness, suppose that there exist another set, Π^*, of probability measures on S and a set, $\mathcal{V}^*$, of mixture continuous, strictly monotonic, utility functions that jointly represent the preference relation $\succ$, where Π^* is distinct from Π and $\mathcal{V}^*$ may or may not be distinct from $\mathcal{V}$. This supposition implies that there exist $\pi^* \in \Pi^* \backslash \Pi$ or $\pi \in \Pi \backslash \Pi^*$.

Since Π is a convex set, if $\pi^* \in \Pi^* \backslash \Pi$ then there exist $s \in S$ such that $\pi^*(s) > \pi(s)$, for all $\pi \in \Pi$. Hence, there is ρ such that $\pi^*(s) > \rho > \pi(s)$, for all $\pi \in \Pi$. Consider that act $\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}}$. By the argument in the proof of Lemma 2, $\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}} \asymp \pi(s)\delta^{\bar{x}} + (1-\pi(s))\delta^{\underline{x}}$, for all $\pi \in \Pi$. By (A.3), $\rho\delta^{\bar{x}} + (1-\rho)\delta^{\underline{x}} \succ \pi(s)\delta^{\bar{x}} + (1-\pi(s))\delta^{\underline{x}}$, for all $\pi \in \Pi$. Hence, by (A.6), $\rho\delta^{\bar{x}} + (1-\rho)\delta^{\underline{x}} \succ \delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}}$.

Since the lottery $\pi^*(s)\delta^{\bar{x}} + (1-\pi^*(s))\delta^{\underline{x}}$ strictly first-order stochastically dominates the lottery $\rho\delta^{\bar{x}} + (1-\rho)\delta^{\underline{x}}$, by strict monotonicity of V^*,

$$V^*\left(\pi^*(s)\delta^{\bar{x}} + (1-\pi^*(s))\delta^{\underline{x}}\right) > V^*\left(\rho\delta^{\bar{x}} + (1-\rho)\delta^{\underline{x}}\right), \ \forall V^* \in \mathcal{V}^*. \tag{11}$$

But $\pi^* \in \Pi^*$, implies that $\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}} \asymp \pi^*(s)\delta^{\bar{x}} + (1-\pi^*(s))\delta^{\underline{x}}$. Hence, it is not true that $V^*(\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}}) < V^*\left(\pi^*(s)\delta^{\bar{x}} + (1-\pi^*(s))\delta^{\underline{x}}\right)$, $\forall V^* \in \mathcal{V}^*$. Thus, for some $V^* \in \mathcal{V}^*$, $V^*(\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}}) \geq V^*\left(\pi^*(s)\delta^{\bar{x}} + (1-\pi^*(s))\delta^{\underline{x}}\right)$. But $\rho\delta^{\bar{x}} + (1-\rho)\delta^{\underline{x}} \succ \delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}}$ implies $V^*(\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}}) < V^*\left(\rho\delta^{\bar{x}} + (1-\rho)\delta^{\underline{x}}\right)$, for all $V^* \in \mathcal{V}^*$. Hence, (11) implies that $V^*(\delta^{\bar{x}}_{\{s\}}\delta^{\underline{x}}) < V^*\left(\pi^*(s)\delta^{\bar{x}} + (1-\pi^*(s))\delta^{\underline{x}}\right)$, $\forall V^* \in \mathcal{V}^*$. A contradiction.

The uniqueness of $\mathcal{V}$ is an implication of Evren and Ok (2011) Remark 1. ■

5.3 Proof of Theorem 3

$(i) \Rightarrow (ii)$. Define a binary relation $\succeq$ on ΔX by $p \succeq q$ if $\neg(q \succ p)$, for all $p, q \in \Delta X$. Axioms (A.1) and (A.6) imply that $\succeq$ is complete and transitive. Denote by $\sim$ the symmetric part of $\succeq$. For each $p \in \Delta X$ define v_p by $p \sim v_p \delta^{\bar{x}} + (1 - v_p) \delta^{\underline{x}}$. Define a function $V : \Delta X \to [0,1]$ by $V(p) = v_p$. By (A.1) - (A.3) V is well-defined, mixture continuos, strictly monotonic function on ΔX, and $p \succeq q$ if and only if $V(p) \geq V(q)$.

By Theorem 1, there exists a unique set, Π, of probability measures on S such that, for all $f \in H$ and $\pi \in \Pi$, $f \asymp \Sigma_{i=1}^{n} \pi(s_i) f(s_i)$. For each $\pi \in \Pi$, define a function $V^{\pi} : H \to [0,1]$ by $V^{\pi}(f) = V(\Sigma_{i=1}^{n} \pi(s_i) f(s_i))$. Let $\mathcal{V} := \{V^{\pi} \mid \pi \in \Pi\}$. But, by Lemma 1, $f \succ g$ if and only if $V^{\pi}(f) > V^{\pi}(g)$, for all $V^{\pi} \in \mathcal{V}$. Hence, $f \succ g$ if and only if, $V(\Sigma_{i=1}^{n} \pi(s_i) f(s_i)) \succ V(\Sigma_{i=1}^{n} \pi(s_i) g(s_i))$, for all $\pi \in \Pi$.

That $(ii) \Rightarrow (i)$ and the uniqueness of the representation follow from the corresponding parts in the proof of Theorem 1. ■

5.4 Proof of Theorem 4

$(i) \Rightarrow (ii)$. By (A.3), (A.8) and the argument in the proof of Lemma 2, Π is a singleton set. Thus, by Theorem 1, for all $f, g \in H$, $f \succ g \Leftrightarrow V(\Sigma_{s \in S} \pi(s) f(s)) > V(\Sigma_{s \in S} \pi(s) g(s))$, $\forall V \in \mathcal{V}$, where $\mathcal{V}$ is a set of mixture continuos, strictly monotonic, real-valued functions on ΔX.

$(ii) \Rightarrow (i)$. That the axioms (A.1) - (A.6) are implied by the presentation follows form Theorem 1. To show that (A.8) holds, let $E \subseteq S$. By strict monotonicity and mixture continuity of V, there is a unique α^* such that $V\left(\alpha^* \delta^{\bar{x}} + (1 - \alpha^*) \delta^{\underline{x}}\right) = V\left(\delta_E^{\bar{x}} \delta^{\underline{x}}\right)$, for all $V \in \mathcal{V}$. Thus, if $\alpha > \alpha^*$ then $V\left(\alpha \delta^{\bar{x}} + (1 - \alpha) \delta^{\underline{x}}\right) > V\left(\delta_E^{\bar{x}} \delta^{\underline{x}}\right)$ and $V\left(\alpha \delta^{\bar{x}} + (1 - \alpha) \delta^{\underline{x}}\right) \leq V\left(\delta_E^{\bar{x}} \delta^{\underline{x}}\right)$, otherwise. Hence, by (A.3) and strict monotonicity of V, $V\left(\alpha' \delta^{\bar{x}} + (1 - \alpha') \delta^{\underline{x}}\right) < V\left(\delta_E^{\bar{x}} \delta^{\underline{x}}\right)$ for all $\alpha^* > \alpha'$ and $V \in \mathcal{V}$. Thus, for each $E \subseteq S$, $\alpha \delta^{\bar{x}} + (1 - \alpha) \delta^{\underline{x}} \succ \delta_E^{\bar{x}} \delta^{\underline{x}}$ or $\delta_E^{\bar{x}} \delta^{\underline{x}} \succ \alpha' \delta^{\bar{x}} + (1 - \alpha') \delta^{\underline{x}}$, for all $\alpha > \alpha'$.

The uniqueness follows from Theorem 1. ■

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