Managing Credit Booms and Busts: A Pigouvian Taxation Approach*

Olivier Jeanne
Johns Hopkins University

Anton Korinek
University of Maryland

This Version: January 2012
First Version: October 2009

Abstract

We study a dynamic model in which the interaction between debt accumulation and asset prices magnifies credit booms and busts. Borrowers do not internalize these feedback effects and therefore suffer from excessively large booms and busts in both credit flows and asset prices. We show that a Pigouvian tax on borrowing may induce borrowers to internalize these externalities and increase welfare. We calibrate the model by reference to (i) the US small and medium-sized enterprise sector and (ii) the household sector, and find the optimal tax to be countercyclical in both cases, dropping to zero in busts and rising to approximately half a percentage point of the amount of debt outstanding during booms.

JEL Codes: E44, G38, G01
Keywords: boom-bust cycles, financial crises, pecuniary externalities, precautionary savings, macro-prudential regulation

1 Introduction

The interaction between debt accumulation and asset prices contributes to magnify the impact of booms and busts. Increases in borrowing and in collateral prices feed each other during booms. In busts, the feedback turns negative, with credit constraints leading to asset price declines and further tightening of

*The authors would like to thank the *Fondation Banque de France* for financial support. We would also like to thank Marco Battaglini, Sudipto Bhattacharya, David Cook, Dean Corbae, C. Bora Durdu, Tim Fuerst, Hugo Hopenhayn, Nobuhiro Kiyotaki, Alberto Martin, Enrique Mendoza, Raoul Minetti, Joseph Stiglitz and Dimitri Vayanos as well as participants of seminars at the Fed Board, Goethe University Frankfurt, Johns Hopkins, Harvard, Michigan, MIT, UNC Chapel Hill, the 2011 AEA Meetings, the Spring 2011 NBER IFM Program Meeting, and of conferences at the Dallas Fed, the ECB, the Einaudi Institute, the IMF, Tilburg University, Toulouse and the Paul Woolley Center for helpful comments and suggestions. Financial support from the Europlace Institute of Finance is gratefully acknowledged.

credit. This type of mechanism has received a lot of attention following the US financial crisis, and it has been suggested that prudential policies could be used to mitigate the build-up in systemic vulnerability during the boom (Shleifer and Vishny, 2011).

This paper makes a step toward formalizing such policies in a dynamic optimizing model of consumption-based asset pricing and collateralized borrowing. Our model is stripped down to the essence of the mechanism that we want to study. We consider a group of borrowers who enjoy a comparative advantage in holding an asset and who can use this asset as collateral. Their borrowing capacity is therefore increasing in the price of the asset. The asset price, in turn, is driven by their aggregate borrowing capacity. This introduces a mutual feedback loop between asset prices and credit flows, so that small shocks may be amplified and lead to large simultaneous booms or busts in asset prices and credit flows.

The model attempts to capture, in a stylized way, a number of economic settings in which the systemic interaction between credit and asset prices may be important. The borrowers could be interpreted as a group of entrepreneurs who have more expertise than outsiders to operate a productive asset, or as households putting a premium on owning durable consumer assets or their homes. Alternatively, the borrowers could represent a group of investors who enjoy an advantage in dealing with a certain class of financial assets, for example because of superior information or superior risk management skills.[1] One advantage of studying these situations with a common framework is to bring out the commonality of the problems and of the required policy responses.

The free market equilibrium is constrained inefficient. The asset-debt loop entails a pecuniary externality that leads borrowers to undervalue the benefits of conserving liquidity as a precaution against busts. A borrower who has one more dollar of liquid net worth when the economy experiences a bust relaxes not only his private borrowing constraint but also the borrowing constraints of all other borrowers. Not internalizing this spillover effect, borrowers take on too much debt during good times. As a result, it is optimal to impose a cyclical tax on debt to prevent borrowers from taking on socially excessive levels of debt—something reminiscent of the "macroprudential policies" that have been discussed in recent policy debates.[2]

We explore the quantitative implications of the model based on the experience of different real sectors of the US economy in the 2008-09 crisis. If the model is calibrated to the US small and medium-sized enterprise (SME) sector, we find that the optimal tax converges to 0.56 percent of the amount of debt outstanding over the course of a boom, and drops to zero when a bust oc-

[1] The borrowers could also be interpreted as the residents of a country who borrow from foreign investors. However, the open economy interpretation of our model is limited by the fact that we have not introduced an exchange rate.

[2] The term "macroprudential" is generally used in the context of banking regulation, the objective of macroprudential regulation being to take a more systemic perspective on bank risk management than traditional "microprudential" regulation (see e.g. Borio, 2003). We use the term "macroprudential" in a broader sense, since it is not restricted to the credit flows that are intermediated by banks.

curs. Borrowing by the US household sector is subject to externalities of similar magnitude.

Even though our framework is stylized, we find that the optimal macroprudential tax on borrowing is a non-trivial function of the environment. The optimal macroprudential tax on borrowing may respond to changes in parameter values in surprising ways. For example, an increase in the probability of a bust may call for *lower* macroprudential taxation. This is because a riskier environment may increase private self-insurance sufficiently that there is less need for public intervention. Furthermore, there are parameter configurations for which optimal macroprudential taxation does not affect the equilibrium when debt exceeds a threshold because the tax would have to be excessively high to relax the collateral constraint.

We study three extensions of the basic model and find that its essential properties are preserved. First, we change the nature of the shock by assuming that it affects the availability of credit rather than the income of borrowers. Then we look at the case where borrowers can issue long-term debt or equity. All three of these extensions change some features of the boom-bust cycle equilibrium, but it remains true that the constrained optimum can be achieved by a cyclical tax on debt, and this tax is of the same order of magnitude as in the benchmark model.

Literature Our model is related to the positive study of financial amplification effects in closed and open economy macroeconomics. Bernanke and Gertler (1989) and Greenwald and Stiglitz (1993) show analytically that financial imperfections may amplify the response of an economy to fundamental shocks. Kiyotaki and Moore (1997) investigate the scope for amplification in a model of credit constraints that depends on borrowers' ability to commit to repay based on the *future* value of their collateral assets. Kocherlakota (2000) and Cordoba and Ripoll (2004) show that the magnitude of amplification effect is likely to be low in a realistic calibration of such a model because the impact of current shocks on the future value of collateral is mitigated by increased consumption of borrowers, increased investment and rising interest rates.

By contrast, we use a model of credit constraints that depend on borrowers' ability to renegotiate based on the *current* value of their collateral assets, which may be depressed for liquidity reasons. We show that this mechanism delivers quantitatively significant amplification effects. Papers that study similar constraints from a positive perspective in the literature on emerging market business cycles include Mendoza and Smith (2006) and Mendoza (2010). Bernanke, Gertler and Gilchrist (1999) present an alternative mechanism of amplification based on endogenous changes in external finance premia.

Our paper is also related to analyses of the ongoing world-wide credit crisis that emphasize the amplifying mechanisms involving asset price deflation and deleveraging in the financial sector (e.g., Brunnermeier, 2009; Adrian and Shin, 2010).

On the normative side, Gromb and Vayanos (2002), Caballero and Krishnamurthy (2003), Lorenzoni (2008), Korinek (2009, 2010), Jeanne and Korinek

(2010a) and Stein (2011) have analyzed the externalities of financial amplification in stylized three-period models. By contrast, this paper considers an infinite-horizon setup, which allows us to study macroprudential policies over booms and busts and to give a more quantitative flavor to the analysis. This is particularly relevant for determining the optimal magnitude of regulatory measures in practice. Bianchi (2011) considers a quantitative model of an emerging market economy in which real exchange rate depreciations may give rise to financial amplification and characterizes optimal policy responses. By contrast, our paper studies financial amplification in a setting in which the deleveraging externality involves an asset price rather than the real exchange rate, which is applicable to both industrialized and emerging economies.

Bianchi and Mendoza (2010) consider a model where credit constraints that depend on asset prices affect labor demand. They assume that households produce a consumption good using the labor of other households and must borrow a fraction of the wage bill, so that the collateral constraint reduces the effective demand for labor and the supply of consumption good. By contrast, we focus on a pure endowment economy to study the fundamental driving forces involved in financial amplification – that falling asset prices tighten the constraints on already constrained borrowers. The welfare cost of the credit constraint in our framework consists of deviations from consumption smoothing—a cost that is more generic in the sense that it mirrors the costs of financial amplification in any economy with borrowers who have a concave payoff function over the amount borrowed. The optimal Pigouvian tax on debt in calibrated versions of our model is somewhat lower but of the same order of magnitude as in Bianchi and Mendoza (2010), suggesting that mitigating booms and busts in consumption is, in and of itself, a significant determinant of such taxes.

While our paper studies optimal macro-prudential regulation to reduce the cost of financial crises ex-ante, Benigno et al. (2010, 2011) show that there is also scope for ex-post intervention in the event of binding constraints if a planner has a policy tool to affect sectoral labor supply. They show furthermore that effective ex-post interventions may allow borrowers to take on a larger quantity of debt ex-ante by reducing the necessity for precautionary savings. Some have interpreted this to suggest that macroprudential restrictions on borrowing are undesirable. However, we show in Jeanne and Korinek (2012) that the optimum of a planner who has access to ex-ante and ex-post instruments of the type studied by Benigno et al. is characterized by a positive macroprudential tax on borrowing and may still exhibit a higher equilibrium quantity of debt.

Nikolov (2010) examines the scope for limits on leverage in a closed economy akin to Kiyotaki and Moore (1997). He calibrates the model in such a way that the most productive agents are always financially constrainted. Nikolov shows that if financial constraints are sufficiently tight, then a planner does not find it worthwhile to reduce debt, i.e. the planner chooses the same allocations (determined by the binding constraint) as the decentralized equilibrium. In our sensitivity analysis, we replicate this result for certain parameter values, but also observe that there is in general scope for macroprudential taxation.

Finally, our paper presents a numerical solution method for DSGE models with occasionally binding endogenous constraints that extends the endogenous gridpoints method of Carroll (2006). This method allows us to solve such models in an efficient way and may enable researchers to analyze more complex models than what has been computationally feasible in the existing DSGE literature with endogenous constraints, ultimately producing policy guidance on richer and more realistic models of the economy. We also point to the possibility of multiple equilibria in DSGE models of financial amplification (self-fulfilling crashes in asset prices).

The structure of the paper is as follows. Section 2 presents the assumptions of the model. Section 3 compares the laissez-faire equilibrium with a social planner. Section 4 presents a calibration of our model and explores its quantitative implications. Section 5 discusses extensions of the benchmark model, and section 6 concludes.

2 The model

2.1 Setup

We consider a group of identical atomistic individuals, indexed by $i \in [0,1]$, in infinite discrete time $t = 0, 1, 2, ...$ The utility of individual i at time t is given by,

$$U_{i,t} = E_t \left(\Sigma_{s=t}^{+\infty} \beta^{s-t} u(c_{i,s}) \right), \tag{1}$$

where $\beta < 1$ is a discount factor and the utility function has constant relative risk aversion,

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

We will consider equilibria in which individuals are impatient and borrow—thus we call them "borrowers." They receive two kinds of income, the payoff of an asset that can serve as collateral, and an endowment income. Borrower i maximizes his utility under the budget constraint

$$c_{i,t} + a_{i,t+1} p_t + \frac{w_{i,t+1}}{R} = (1-\alpha) y_t + a_{i,t} \left(p_t + \alpha y_t \right) + w_{i,t}, \tag{2}$$

where $a_{i,t}$ is the borrower's holdings of the collateral asset at the beginning of period t and p_t is its price; $w_{i,t}$ is his financial wealth in the form of bond holdings at the start of period t; y_t is total income in period t and is the same for all borrowers, and αy_t is the share of that income that comes from the asset.[3] Since he is impatient, the representative borrower's wealth w will be negative in equilibrium and we can call $-w$ his debt. The representative borrower's debt is held by outside lenders who have an indefinite demand/supply for risk-free bonds at the safe interest rate $r = R - 1$.

[3] If the collateral asset were productive capital used in a Cobb-Douglas production function, and labor were provided by outsiders, α would be the exponent of capital in the production function.

Total income y_t follows a stochastic process which, for the sake of simplicity, we assume to be independent and identically distributed, although it would be straightforward to extend the analysis to the case where it is Markov. Assuming an i.i.d. process for y_t is not too restrictive given that, in the calibration, we will consider shocks that represent relatively rare disasters (like in Barro, 2009) rather than business cycle fluctuations.[4]

The collateral asset is not reproducible and the available stock of asset is normalized to 1. The asset can be exchanged between borrowers in a perfectly competitive market, but we do not allow them to sell the asset to outside lenders and rent it back because borrowers derive important benefits from the control rights that ownership provides. For simplicity we assume that the asset would become worthless if it was sold to outside lenders. Therefore $a_{i,t}$ must be equal to 1 in a symmetric equilibrium. This assumption can be relaxed to some extent (see section 4.3), but we need some restriction on asset sales so that borrowers issue collateralized debt in equilibrium.

Furthermore, we assume that the only financial instrument that can be traded between borrowers and lenders is uncontingent one-period debt. The assumption that debt is uncontingent can be justified e.g. on the basis that shocks to borrowers are not verifiable and cannot be used to condition payments.[5] The feature that debt is short-term provides borrowers with adequate incentives (we present an extension to long-term debt in section 4.2). Both assumptions correspond to common practice across a wide range of financial relationships.

The amount of debt that the borrowers can roll over is limited by a collateral constraint. The microfoundation for the collateral constraint is similar to that in Kiyotaki and Moore (1997). We assume that after rolling over his debt in period t, a borrower can renegotiate the level of his debt. If the negotiation fails, the creditors receive a fixed amount of good ψ plus a certain quantity of collateral asset. The borrower has all the bargaining power, and so negotiates his debt down to the amount that the creditors can recover in a default. In order to discourage renegotiation, the level of debt must satisfy the collateral constraint,

$$-\frac{w_{i,t+1}}{R} \leq \psi + f(a_{i,t+1})p_t, \tag{3}$$

where $f(a) \leq a$, the quantity of asset that creditors can seize in a default, is an increasing function of the borrower's asset holding a. In general, the shape of function $f(\cdot)$ depends on legal rights of lenders as well as the borrower's ability to hide or abscond with his assets. We will assume that a lender can seize all the borrower's collateral up to a level $\phi \leq 1$, that is,

$$f(a) = \min(a, \phi). \tag{4}$$

[4] We could also introduce growth into the model. The model with growth, once detrended, would be isomorphic to the model presented here.

[5] The findings of Korinek (2010) suggest that our results on excessive exposure to binding constraints would continue to hold when borrowers have access to costly state-contingent financial contracts.

Function $f(\cdot)$ could be specified in other ways. For example, one could assume that creditors can seize a fraction ϕ of the collateral asset, i.e., $f(a) = \phi a$. We present the equilibrium conditions resulting from this alternative approach in appendix A.2 and discuss why the qualitative and quantitative implications of the two specifications are almost identical. We prefer to use specification (4) because it presents some advantages in terms of tractability, in particular in our extensions in section 5. (In a symmetric equilibrium, $a = 1$ and the specification in the constraint (4) implies that $f = \phi$.)

Note that the borrower can renegotiate his debt right after it is issued but is committed to repay in the following period. This captures the notion that the liquidity of the asset in period t matters. We could allow for renegotiation at the time of repayment too, which would introduce an additional constraint involving p_{t+1} rather than p_t on the right-hand side of (3). The financial amplification dynamics, however, come from the feedback loop between p_t and c_t and would not be significantly altered.[6]

2.2 Equilibrium conditions under laissez-faire

We derive in the appendix the first-order conditions for the optimization problem of a borrower i. We then use the fact that in a symmetric equilibrium, all individuals are identical and hold one unit of collateral asset ($\forall i, t \quad a_{i,t} = 1$). Variables without the subscript i refer to the *representative* borrower (or equivalently, to aggregate levels, since the mass of borrowers is normalized to 1). This gives the following two conditions

$$u'(c_t) = \lambda_t + \beta R E_t \left[u'(c_{t+1}) \right], \tag{5}$$

$$p_t = \frac{\beta E_t \left[u'(c_{t+1})(\alpha y_{t+1} + p_{t+1}) \right]}{u'(c_t)}, \tag{6}$$

where λ_t is the costate variable for the borrowing constraint (3). The first equation is the Euler condition and the second one is the standard asset pricing equation.

The equilibrium is characterized by a set of functions mapping the state of the economy into the endogenous variables. Given that y_t is i.i.d., we can

[6] The literature has explored both forms of collateral constraints to generate financial amplification. In Kiyotaki and Moore (1997), financial amplification arises from a feedback loop between falling borrowing capacity today, falling investment today and falling asset prices tomorrow. This requires incorporating asset investment in the analysis and introduces an additional state variable into the problem. In an endowment economy, a collateral constraint that depends on tomorrow's price does not lead to financial amplification since being borrowing constrained today does not directly affect the asset price tomorrow. From a quantitative perspective, Kocherlakota (2000) and Cordoba and Ripoll (2004) suggest that collateral constraints of the type of Kiyotaki and Moore (1997) that rely on the next-period price p_{t+1} do not lead to quantitatively significant financial amplification. By contrast, Mendoza (2010) describes financial amplification as arising from a feedback loop between falling borrowing capacity today and falling asset prices today and obtains quantiatively significant financial amplification.

summarize the state by one variable, the beginning-of-period liquid net wealth (excluding the value of the collateral asset),

$$m_t \equiv y_t + w_t.$$

We do not include the asset in the definition of net wealth because its price, p_t, is an endogenous variable. In a symmetric equilibrium the budget constraint (2) simplifies to

$$c_t + \frac{w_{t+1}}{R} = m_t, \tag{7}$$

and the collateral constraint (3) can be written, in aggregate form,

$$c_t \leq m_t + \psi + \phi p_t. \tag{8}$$

The equilibrium, thus, can be characterized as follows.

Proposition 1 *The laissez-faire equilibrium is characterized by three functions, $c(\cdot)$, $p(\cdot)$ and $\lambda(\cdot)$ that satisfy the first-order conditions,*

$$c(m)^{-\gamma} = \lambda(m) + \beta R E\left[c(m')^{-\gamma}\right], \tag{9}$$

$$\lambda(m) = \left[(m + \psi + \phi p(m))^{-\gamma} - \beta R E\left(c(m')^{-\gamma}\right)\right]^+, \tag{10}$$

$$p(m) = \beta E\left[c(m')^{-\gamma}(\alpha y' + p(m'))\right] c(m)^{\gamma}, \tag{11}$$

where the transition equation for net wealth is

$$m' = y' + R\,(m - c(m)). \tag{12}$$

Proof. See discussion above ■

2.3 Social planner

We introduce a planner into the economy who determines borrowing, but does not directly interfere in asset markets—that is, she takes as given that the asset price is determined by the marginal rate of substitution between assets and consumption goods, equation (6). This corresponds to a setup in which the planner can choose allocations subject to the same constraints as borrowers, but has no additional instruments.[7] An alternative interpretation for our planning setup is that borrowers coordinate to internalize the externalities that they impose on each other.

In period t, the planner makes the consumption/savings decision of the representative borrower and determines savings, w_{t+1}, before the asset market opens

[7] If we grant additional instruments to the planner, she can improve the equilibrium further. In particular, if the planner has an instrument to costlessly move asset prices, she will always be able to relax the financial constraint and replicate the first-best allocation in the economy. For a detailed discussion, see Jeanne and Korinek (2011).

at time t. We look for time-consistent equilibria in which the social planner optimizes on w_{t+1} taking the future policy functions $c(m)$ and $p(m)$ as given. (Although we do not change the notation, those policy functions are not the same as in the laissez-faire equilibrium.) Assuming that the policy functions $c(m)$, $p(m)$ and $\lambda(m)$ apply in the following period, equation (6) implicitly defines the asset price as a function of the state and of current consumption,

$$\hat{p}(m,c) = \beta E\left[u'(c(m'))(\alpha y' + p(m'))\right]/u'(c), \tag{13}$$

where the expectation is taken conditional on m and c, with $m' = y' + R(m-c)$. The social planner affects the asset price indirectly via his choice of consumption c versus savings w'. Since borrowers are still subject to the collateral constraint (3)-(4), the planner, taking $a_t = 1$ as given, sets w' subject to

$$-\frac{w'}{R} \leq \psi + \phi\hat{p}\left(m, m - w'/R\right). \tag{14}$$

We can therefore formulate the planner's optimization problem as solving the same problem as borrowers, but subject to the borrowing constraint (14). We make the following assumption:

Assumption A1. *The price function $\hat{p}(m,c)$ is differentiable and satisfies $\phi \cdot \frac{\partial \hat{p}(m,c)}{\partial c} < 1$ so that the left-hand-side of (14) is increasing with w'.*

This assumption guarantees that when the social planner reduces aggregate debt (increases w'/R), the collateral constraint (14) is relaxed. By implication, if a certain level of debt $-w'$ is consistent with the social planner's collateral constraint (14), then any lower level of debt is also consistent with the constraint. This is not necessarily the case because reducing aggregate debt lowers the price of the collateral. However, the assumption is satisfied if ϕ is sufficiently small, so that the variations of the left-hand-side of equation (14) are dominated by the first term, w'/R.[8]

Conditional on assumption A1, the first-order conditions for the social planner's problem are given in the following Proposition.

Proposition 2 *The social planner equilibrium is characterized by three functions, $c(\cdot)$, $p(\cdot)$ and $\lambda(\cdot)$ that are given by equations (10), (11) and*

$$c(m)^{-\gamma} = \lambda(m) + \beta R E\left[c(m')^{-\gamma} + \phi\lambda(m')p'(m')\right]. \tag{15}$$

where $p'(m')$ is the first-derivative of the next-period asset price with respect to the next-period aggregate liquid net wealth.

[8] As we discuss below in section 3.1, the assumption is also related to the uniqueness of equilibrium in our model. In appendix A.4 we discuss how different parameter values affect the inequality in a simplified version of the model that can be solved analytically.

Proof. Assumption A1 implies that w' must be above a minimum level. Let us denote by $\bar{p}(m)$ the price of the asset when the constraint is binding. Then the planner's credit constraint can be rewritten,

$$\frac{w'}{R} + \psi + \phi\bar{p}(m) \geq 0. \tag{16}$$

The constrained social planner maximizes the utility of the representative borrower subject to the budget constraint $w'/R + c = m$ and to the credit constraint (16). The first-order condition for consumption and saving is

$$u'(c_t) = \lambda_t + \beta R E_t \left[u'(c_{t+1}) + \phi\lambda_{t+1}\bar{p}'(m_{t+1}) \right]. \tag{17}$$

Note that $\bar{p}(m)$ is not the same as the equilibrium price $p(m)$. The equilibrium price $p(m)$ is not the same as $\bar{p}(m)$ if the economy is unconstrained, but then λ_{t+1}in the equation above is equal to zero and the term drops out. On the other hand, $p(m)$ it is equal to $\bar{p}(m)$ if the economy is constrained (equation (16) is binding). Since $p(m_{t+1}) = \bar{p}(m_{t+1})$ if $\lambda_{t+1} > 0$, it follows that $\bar{p}'(m_{t+1})$ can be replaced by $p'(m_{t+1})$ in the first-order condition (17). This implies equation (15). ■

By comparing the Euler equations (9) and (15), one can see that if the price of the asset is increasing with aggregate net liquid wealth, $p'(\cdot) > 0$, the social planner raises saving above the laissez-faire level, strictly so if there is a risk that the collateral constraint will bind in the next period. The planner's wedge is proportional to the expected product of the shadow cost of the credit constraint times the derivative of the debt ceiling with respect to wealth. The planner internalizes that increasing aggregate savings today raises tomorrow's asset price and relaxes tomorrow's credit constraint.

Under the planner, precautionary savings is augmented by a systemic component: the social planner implements a policy of *macro-prudential saving*. This does not come from the fact that the planner estimates risks better than individuals. Decentralized agents are aware of the risk of credit crunch and maintain a certain amount of precautionary saving (they issue less debt than if this risk were absent). But they do not internalize the contribution of their precautionary savings to reducing the *systemic* risk coming from the debt-asset deflation spiral.

Equation (15) illustrates that the planner's motive is purely prudential and forward-looking. In particular, she does not reduce saving so as to increase the price of collateral and relax the credit constraint in the current period. Either the economy is constrained, and it is impossible to reduce saving, or the economy is unconstrained and there is no benefit from relaxing the constraint.

The planner's Euler equation provides guidance for how the constrained optimal equilibrium can be implemented via taxes on borrowing. Decentralized borrowers undervalue the social cost of debt by the term $\phi E\left[\lambda(m')p'(m')\right]$ on the right-hand side of the planner's Euler equation (15). The planner's equilibrium can be implemented by a Pigouvian tax $\tau_t = \tau(m_t)$ on borrowing that is rebated as a lump sum transfer $T_t = -\tau_t w_{t+1}/R$:

$$c_t = y_t + w_t - \frac{w_{t+1}}{R}(1-\tau_t) + T_t. \tag{18}$$

The tax introduces a wedge in the borrowers' Euler equation,

$$(1-\tau_t)\,u'(c_t) = \lambda_t + \beta R E_t\left[u'(c_{t+1})\right],$$

and replicates the constrained social optimum if it is set to

$$\tau(m) = \frac{\phi\beta R E_t\left[\lambda(m')p'(m')\right]}{u'(c(m))}, \tag{19}$$

where all variables are evaluated at the social optimum.

3 Quantitative Exploration

We now turn the attention to the quantitative implications of the model. Sections 3.1 and 3.2 respectively present our numerical solution method and our calibration. Section 3.3 discusses the results of a numerical simulation with booms and busts in the asset price and in credit flows. The last section presents some sensitivity analysis.

3.1 Numerical solution

In order to generate a persistent motive for borrowing, we need to assume that borrowers are impatient relative to outsiders, i.e.,

$$\beta R < 1.$$

We can make conjectures about the form of the solution by analogy with the case with an exogenous credit constraint studied for example in Carroll (2008). We consider equilibria in which the consumption function $m \mapsto c(m)$ is a continuously increasing function of wealth. Let us denote by $\underline{m}$ the level of wealth for which consumption is equal to zero,

$$c(\underline{m}) = 0.$$

By analogy with the case with an exogenous credit constraint, we would expect the borrowers to be credit-constrained in a wealth interval $m \in [\underline{m}, \bar{m}]$, and to be unconstrained for $m \geq \bar{m}$. It is not difficult to see that the lower threshold must be equal to

$$\underline{m} = -\psi.$$

This results from the facts that $c(m) \leq m + \psi + \phi p$, and that p converges to zero as c goes to zero (by equation (6)). The upper threshold, $\bar{m}$, above which borrowers are unconstrained must be determined numerically.

The numerical solution method is an extension of the endogenous grid points method of Carroll (2006) to the case where the credit constraint is endogenous. The procedure performs backwards time iteration on the agent's optimality conditions. We define a grid $\mathbf{w}$ for next period wealth levels w' and combine the next period policy functions with agent's optimality conditions to obtain current period policy functions until the resulting functions converge. The difference with Carroll (2006) is that the threshold level at which the borrowing constraint becomes binding is endogenous. This implies that the minimum level of wealth is itself a function of the state, which is obtained by iterating on the asset pricing equation (11). The details of the numerical solution method are provided in the appendix.

One issue in the implementation of our numerical method is that it does not converge if ϕ exceeds a threshold, that is, if the credit constraint is too sensitive to the price of the collateral. We strongly suspect that our numerical algorithm does not converge when assumption A.1 is violated and the model exhibits multiple equilibria. As we show in appendix A.4, the self-reinforcing loop that links consumption to the price of the collateral may lead to self-fulfilling crashes in the price of collateral. We derive there an explicit formula for the threshold in ϕ that guarantees that assumption A.1 is satisfied and that leads to equilibrium uniqueness in the special case where $\beta R = 1$ and there is no uncertainty. We have observed that in the general stochastic case, our numerical resolution method tends to converge if and only if ϕ is taken below this threshold.

3.2 Calibration

Although the model is very stylized, we attempt to calibrate it in a way that is as realistic as possible by looking at data on US households, small and medium enterprises (SMEs) and corporations.

We assume that the process for income is binomial: total income is high (equal to y_H) with probability $1 - \pi$, or low (equal to y_L) with probability π. The high state is the normal state that prevails most of the time, whereas the realization of the low state is associated with a bust in the asset price and in credit, which occurs infrequently. Thus, we calibrate our model by reference to rare and large events rather than real business cycle fluctuations. We will assume that a bust occurs once every twenty years on average.

Our benchmark calibration is reported in Table 1. The riskless real interest rate is set to 3 percent. The discount factor is set to 0.96, a value that is low enough to induce the borrowers to borrow and expose themselves to the risk of a credit crunch. The risk aversion parameter is equal to 2, a standard value in the literature.

Table 1. Benchmark calibration

β	R	γ	α	y_L	y_H	π	ϕ	ψ
0.96	1.03	2	0.2	0.969	1	0.05	0.046	1.97

The other parameters have been calibrated by reference to the experience of the US small and medium enterprises (SMEs) in 2008-09. We have also looked at other US sectors (the household sector and the nonfinancial corporate sector) in order to obtain plausible ranges of variation for the parameters.[9]

The relevant data for the US nonfinancial sectors are shown in Table 2.[10] For each sector we report the change in the value of assets and the change in liabilities during a one-year time window centered on the peak of the crisis (the fall of 2008).[11] For households and SMEs we observe that the value of assets and liabilities both fall, consistent with the model. Corporations also experienced a fall in the value of their assets but they were able to slightly increase their outstanding debt by issuing larger amounts of corporate bonds, in spite of a contraction in bank lending. The difference between SMEs and the corporate business sector, thus, is consistent with the notion that the former are more vulnerable than the latter to a credit crunch because they are more dependent on bank lending.

Table 2. Balance sheet data for US Households, SMEs and Corporations (in $bn)

	Assets[12]			Debt		
	2008Q2	2009Q2	Chg.	2008Q2	2009Q2	Chg.
Households	74,273	64,425	-13.3%	14,418	14,116	-2.1%
SMEs	11,865	10,409	-12.3%	5,410	5,343	-1.2%
Corporations	28,579	26,521	-7.2%	13,039	13,597	+4.3%

Table 3 shows our calibration of α, ϕ, ψ, and y_L for the three sectors covered by Table 2 except US corporations. We do not include the US corporate sector because, as mentioned above, its outstanding debt did not fall during the crisis.

[9] The US Flow of Funds do not report the same balance sheet data for the financial sector as for households or the nonfinancial business sectors. We argue in our companion working paper Jeanne and Korinek (2010b) that ϕ is much higher in the financial sector than in the rest of the economy, and that it probably is in the region with multiple equilibria. It would be interesting to study the quantitative properties of the model when there are multiple equilibria but this is left for future research.

[10] The source is the Federal Reserve's Flow of Funds database in 2010. The data for Households, SMEs and Corporations respectively come from Table B.100 (Households and Nonprofit Organizations) lines 1 and 31, Table B.102 (Nonfarm Nonfinancial Corporate Business) lines 1 and 21, and Table B.103 (Nonfarm Noncorporate Business) lines 1 and 23. The nonfarm noncorporate business sector comprises partnerships and limited liability companies, sole proprietorships and individuals who receive rental income. This sector is often thought to be composed of small firms, although some of the partnerships included in the sector are large companies. More importantly for our purpose, firms in the nonfarm noncorporate business sector generally do not have access to capital markets and, to a great extent, rely for their funding on loans from commercial banks and other credit providers as well as on trade credit.

[11] Although it would not be difficult to adjust those numbers for inflation, this would not change the results if we used the same deflator for assets and liabilities. In addition, the inflation rate was relatively low during this period.

[12] Real estate and equity investments are accounted for at market value; durable assets are accounted for at replacement value; fixed income securities are accounted for at book value in the Fed's Flow of Funds accounts.

The share of the asset in income, α, was inferred from the ratio of the asset price to total income. Abstracting from the risk of bust, the price of the asset converges to $p = \beta/(1-\beta)\alpha y$ in the high state so that[13]

$$\alpha = \frac{1-\beta}{\beta}\frac{p}{y}. \tag{20}$$

The ratio p/y was proxied by taking the ratio of households' asset holdings to national income in the case of households, and the ratio of assets to value added in the case of SMEs.[14] Note that at 20 percent, our estimate of the share of capital in the value added of SMEs is smaller than the share of capital income in total GDP, which is closer to 30 percent. This may reflect the fact that SMEs are less capital intensive than large corporations, or that a larger share of labor income goes to self-employed entrepreneurs.

Table 3. Parameter values for US households and SMEs

	α	ϕ	ψ	y_L
US households	24.5%	3.1%	307%	0.963
US SMEs	20.0%	4.6%	197%	0.969

The two parameters in the collateral constraint, ψ and ϕ, were calibrated using the information in Table 2. The value of ϕ was estimated by dividing the fall in debt by the fall in asset value between the second quarter of 2008 and the second quarter of 2009. Analytically, this can be seen by differencing the borrowing constraint (3) to obtain

$$\phi = \frac{-\Delta w'}{R\Delta p}$$

The resulting values for ϕ are low. In particular, they are much lower than suggested by the microeconomic evidence on the maximum amount of collateral asset that creditors can seize in a default (which is typically close to 100 percent for many real assets). The measure of ϕ that is relevant for the purpose of calibrating our model, however, is the responsiveness of the debt constraint to a fall in the price of collateral inside a given period. This could be much lower than the share of the asset that serves as collateral, for example because debt has a maturity longer than one period and does not respond instantaneously to a fall in the price of collateral. We investigate an extension of the model with long-term debt in section 4.2.

Abstracting from the risk of a bust, the ratio of debt to asset value converges to $\psi/p + \phi = \frac{\psi(1-\beta)}{\alpha\beta y} + \phi$, so that the default penalty ψ can be calibrated as

$$\psi = \frac{\alpha\beta y}{1-\beta}\left(\frac{d}{p} - \phi\right), \tag{21}$$

[13] We verified numerically that the asset price during booms in our model is indeed closely approximated by this formula.

[14] The data for national income and the value added of the noncorporate business sector are taken from the Bureau of Economic Analysis' NIPA statistics, table 1.13 (annual data for 2008).

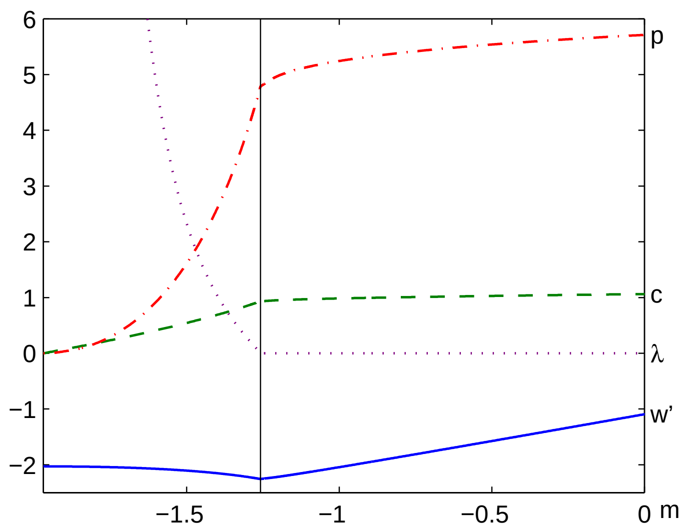

Figure 1: Policy functions $c\,(m)$, $p\,(m)$ and $\lambda\,(m)$.

where d/p is the ratio of debt to asset value. We proxied d/p by taking the ratio of debt to total assets in the second quarter of 2008 in each sector. We then applied formula (21) using the values of α and ϕ derived before and $y = 1$.

Finally, income was normalized to 1 in the high state and y_L was calibrated so as to reproduce the fall in asset value observed in the data in the event of a bust (Table 2).[15]

3.3 Results

Figure 1 shows the policy functions $c(m)$, $p(m)$ and $\lambda(m)$ in the laissez-faire equilibrium for the benchmark calibration in Table 1, which represents the SME sector. The equilibrium is unconstrained if and only if wealth is larger than $\overline{m} = -1.26$. In the unconstrained region, consumption, saving and the price of the asset are all increasing with wealth. Higher wealth raises current consumption relative to future consumption, which bids up the price of the asset.

The levels of consumption and of the asset price vary more steeply with wealth in the constrained region than in the unconstrained region, reflecting the collateral multiplier. Both consumption and the asset price fall to zero when wealth is equal to $-\psi = -1.97$. By contrast, saving w' *decreases* with wealth in the constrained region. Higher wealth is associated with an increase in the price

[15] The price of real estate is determined, in our model, by y_t, which could be interpreted as rental income or as the nonpecuniary utility of home ownership. The latter is not observable and the former did not fall by enough in the recent crisis to explain a 30 percent fall in real estate prices. The recent boom-bust in US real estate may have been to some extent the result of a bubble, which our model does not capture as it does not entail any deviation of the asset price from its fundamental value (conditional on the frictions).

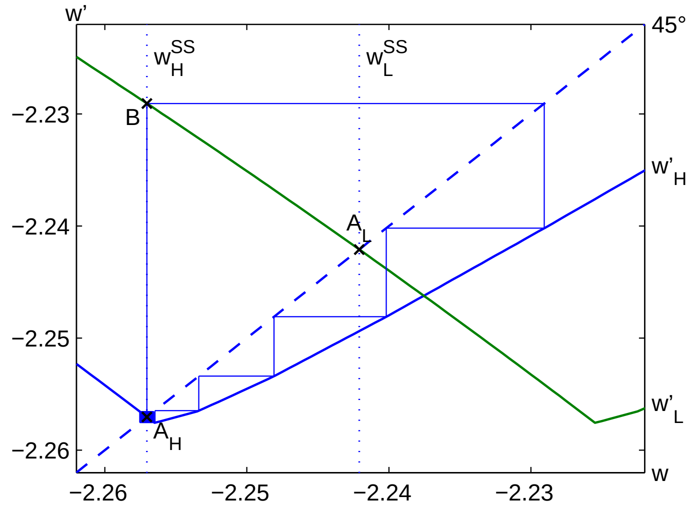

Figure 2: Wealth dynamics

of collateral, which relaxes the borrowing constraint on borrowers and allows them to roll over larger debts.

Figure 2 shows how saving depends on the level of wealth, $w'(w)$, for the two states $y = y_L$, y_H. One can obtain the curve for the low state by shifting the curve for the high state to the right by $\Delta y = y_H - y_L$. The curves intersect the 45^o line in two points, A_H and A_L, which determine the steady state levels of wealth conditional on remaining in each state, respectively denoted by w_H^{SS} and w_L^{SS}. We observe that both A_H and A_L are on the downward-sloping branches of each curve, which means that borrowers borrow to the point where they are financially constrained in both states. Furthermore, borrowers tend to borrow more in the high steady state than in the low steady state ($w_H^{SS} < w_L^{SS}$), which they can do because the price of the collateral asset is higher.

Figure 2 also shows the dynamics of the economy when the steady state is disturbed by a one-period fall in y. At the time of the shock, the economy jumps up from point A_H to point B, as borrowers are forced to reduce their debts by the fall in the price of collateral.[16] The dynamics are then determined by the saving function in the high state (since we have assumed that the low state lasts only one period). The economy converges back to point A_H. As it approaches A_H, wealth follows oscillations of decreasing amplitude (creating what looks like a small black rectangle in the figure). There are oscillations because saving is decreasing with wealth in the constrained regime. There is convergence because the slope of the saving curve is larger than -1 in point A_H for our benchmark calibration. This is not true for any calibration and the equilibrium can exhibit cyclic or chaotic dynamics if ϕ is larger.

[16] The price of the collateral asset falls by 12.3 percent, from 4.81 to 4.22. Thus the borrowing

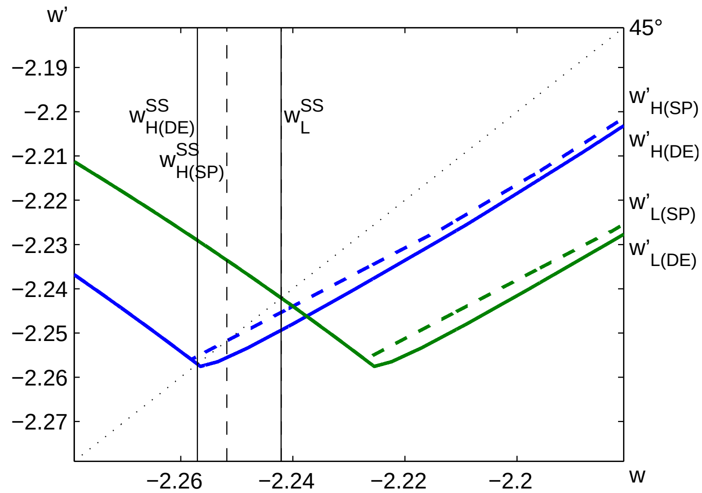

Figure 3: Policy functions of decentralized agents and planner

Figure 3 shows how the social planner (dashed line) increases saving relative to laissez-faire (solid line). The social planner saves more, implying that the economy has a higher level of wealth and it is no longer financially constrained in the high steady state. The $w'(w)$ line is closer to the 45^o line with the social planner, implying that following a one-period fall in y, the economy reaccumulates debt at a lower pace than under laissez-faire.

Finally, Figure 4 illustrates the dynamics of the main variables of interest in the social planner equilibrium with a stochastic simulation. The top panel shows how consumption falls at the same time as output when there is a negative shock. Even with the social planner, consumption falls by more than income because of the fall in the price of collateral. Consumption increases above its long-run level in the period after the shock, when the economy is unconstrained and borrowers inherit low debt from the credit crunch. The same pattern is observed for the price of the collateral.

The bottom panel shows that the optimal Pigouvian tax rate is positive in the high state and zero in the low state.[17] The optimal tax rate in the high steady state is $\tau_H^{SS} = 0.56$ percent. Following equation (19), it is obtained as the product of $\phi = 0.046$, $\pi = 0.05$, $\beta R\lambda \left(m_L^{SS\prime}\right)/u'(c) = 0.134$ and $p'\left(m_L^{SS\prime}\right) = 18$, where $m_L^{SS\prime}$ is the borrower's net wealth in the event that he was in the high steady state in one period and is hit by a bust in the following period.

Note the countercyclical pattern in the tax rate: it falls in a bust, and does not immediately go back to the long-run level after the bust because the

ceiling falls by $\phi \cdot 0.59 \approx 0.03$, which is the distance between A_H and B.

[17] We have set the tax to zero when borrowing is constrained. Any value $\tau \leq \lambda$, including any subsidy to debt $\tau < 0$, would result in the same allocation since the equilibrium is not determined by the Euler equation but by the binding constraint.

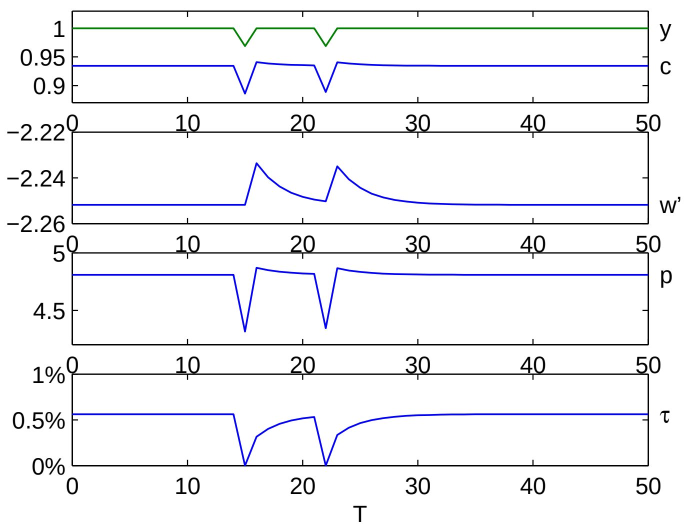

Figure 4: Sample path of y, c, w', p and τ in the planner's equilibrium

economy temporarily has lower debt. The tax rate increases with the economy's vulnerability to a new credit crunch. If the optimal tax rate is imposed on borrowers, the decline in consumption when the economy experiences a bust is reduced from -6.2 percent to -5.2 percent, and the fall in the asset price during a bust is reduced from -12.3 percent to -10.3 percent.

If we calibrate our model using the parameter values reported in Table 3 for the case of the US household sector, the results are similar to what we found for the SME sector. The optimal magnitude of the macroprudential tax in the high steady state is $\tau_H^{SS} = 0.48$ percent for households.

3.4 Sensitivity analysis

We investigate how the optimal Pigouvian taxation depends on the parameters of the economy. Figure 5 shows how τ_H^{SS} (the steady state rate of tax in the high state) varies with the gross interest rate R. For $R = 1.04$, the optimal steady-state tax in the economy is close to zero since $\beta R \approx 1$ and borrowers accumulate a level of precautionary savings that is sufficient to almost entirely avoid debt deflation in case of busts.[18]

As the interest rate declines, it becomes more attractive for borrowers to borrow and the economy becomes more vulnerable to debt deflation in busts. Lower interest rates therefore warrant higher macro-prudential taxation to offset the externalities that individual agents impose on the economy. This effect can be large: the optimal tax rate is multiplied by two when the interest rate is

[18] Recall that we require $\beta R < 1$ for the economy to converge to a stationary equilibrium. If $\beta R \geq 1$, wealth is nonstationary and drifts toward infinity.

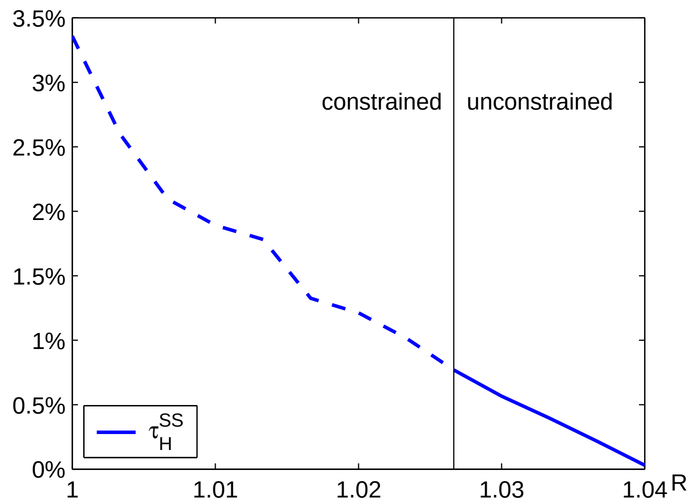

Figure 5: Dependence of macroprudential tax on interest rate

reduced from 2 percent to 1 percent.

For $R \leq 1.026$ (when the line is dashed in Figure 5), the level of debt accumulated by the social planner is high enough that the economy is constrained even in the high steady state A_H. This means that the social planner could lower the tax rate to zero as soon as the economy becomes constrained—although he could also maintain the tax rate at the level shown by the dashed line in Figure 5 or any level in between without changing the equilibrium. In this case, macroprudential taxation matters only in the transition: its role is to slow down the build-up of risk and financial vulnerability after a bust, and thus delay the transition to the constrained regime where the tax no longer matters.

For low levels of R, the strong desire of private agents to borrow creates a dilemma for the social planner. On the one hand, it increases the negative externality associated with debt and so the optimal rate of taxation. On the other hand, it is also costly in terms of welfare not to let private agents take advantage of the low interest rate by borrowing more. In the context of this tradeoff, the social planner may choose to let the economy go into the constrained regime in booms that lasts long enough, i.e., to let aggregate debt be limited by the constraint rather than by the tax.[19]

Figure 6 depicts the response of the steady-state tax rate τ_H^{SS} to changes in parameter ϕ. An increase in ϕ means that the credit constraint becomes more sensitive to the price of collateral. We observe that the optimal tax rate increases with ϕ. A higher ϕ strengthens the potential amplification effects when

[19] This result echoes a point that has often been made by central bankers in the debate on how monetary policy should respond to asset price booms: that the interest rate increase required to discourage agents from borrowing would have to be so drastic to be effective that it is undesirable (see, e.g., Bernanke, 2002).

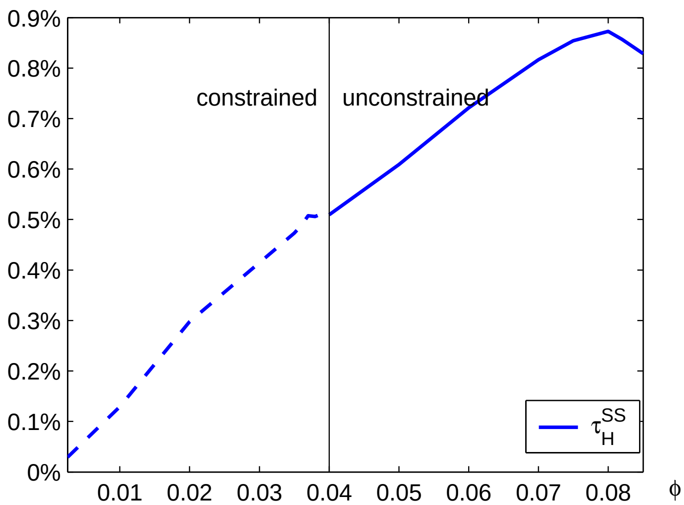

Figure 6: Dependence of macroprudential tax on ϕ

the borrowing constraint becomes binding, and requires tighter macroprudential regulation.

The level of the parameter ϕ also determines whether prudential taxation is transitory or permanent. For low levels of ϕ, we find that the amplification effects are small enough that the planner chooses a constrained equilibrium in steady state and macroprudential taxation is only relevant in the transition from busts to booms. By contrast, if $\phi \geq .037$, the planner implements an unconstrained equilibrium with a Pigouvian tax in steady state. This tax reaches a maximum of almost 1 percent at $\phi = .08$ —for higher values of the parameter the economy becomes so volatile that decentralized agents increase their precautionary savings sufficiently so that the externality declines.[20]

Figures 7 and 8 show how the optimal tax varies with the size and probability of the underlying shock. The optimal tax rate is not very sensitive to those variables: it changes by less than 0.3 percent when the size of the income shock varies between 0 and 10 percent and its probability varies between 0 and 20 percent. The sign of the variation is paradoxical. Figure 7 shows that τ_H^{SS} is increasing with y_L, i.e., the optimal rate of prudential taxation is *decreasing* with the size of the income shock. This result comes from the endogenous response of precautionary savings by private agents to increased riskiness in the economy. As the size of the shock increases, borrowers raise their own precautionary savings, which alleviates the burden on prudential taxation. The tax rate is the highest when the amplitude of the shock is the smallest, but again, these high tax rates do not bind in equilibrium if the income shock is

[20] If ϕ exceeds 0.085 our algorithm does not converge and exhibits oscillations suggestive of multiple equilibria.

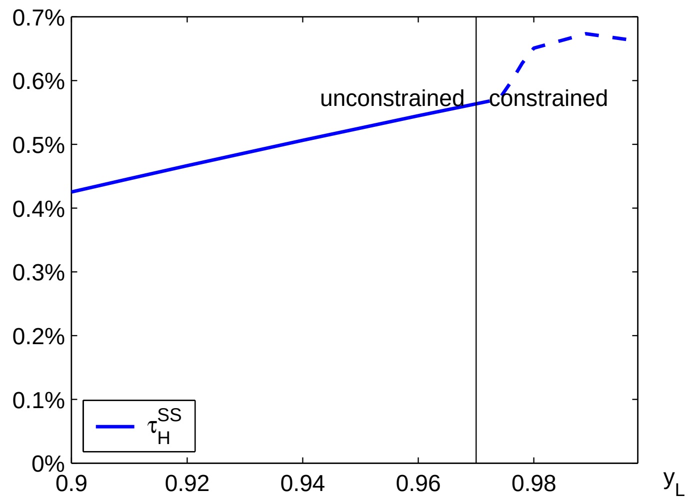

Figure 7: Dependence of macroprudential tax on magnitude of bust $1 - y_L$

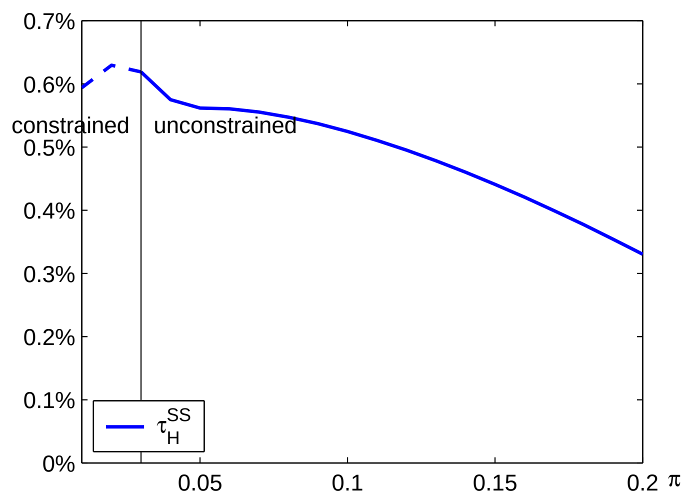

Figure 8: Dependence of macroprudential tax on probability π of bust

very small (below 2 percent).

We observe a similar pattern for the variation of the optimal tax with the probability of a shock (Figure 8). The optimal tax rate is decreasing with π because of the endogenous increase in private precautionary savings. The tax is not binding (and could be set to zero in the long run) if the probability of bust falls below 3 percent. Prudential taxation thus responds the most to "tail risk", i.e., a risk that is realized with a small probability, but not so small that even the social planner can ignore it in the long run. The long-run tax rate is binding and is at its maximum when the probability of shock is between 3 and 5 percent.

4 Extensions

We discuss three extensions of the basic model. In the first one, we assume that the indebted agents are submitted to external financial shocks that abruptly restrict their borrowing constraints. The second and third subsections expand the range of liabilities by assuming that the borrowers can issue equity or long-term debt.

4.1 Fluctuations in Creditworthiness

It has been suggested (see e.g. Jermann and Quadrini, 2010) that the recent global financial crisis was driven more by fluctuations in the availability of credit ("financial shocks") than by developments in the real economy ("endowment shocks"). In our framework, the availability of credit is a function of the parameters ψ and ϕ in the borrowing constraint. We now assume that the economy may be hit by shocks that reduce ψ rather than y.

As in our previous calibration, we choose our parameter values to replicate the declines in credit and asset prices observed during the financial crisis of 2008/09. Income is now deterministic and equal to $y = 1$. The parameter ψ_H is calibrated so as to reproduce the pre-crisis debt-to-income ratio, and ψ_L is calibrated to match the observed fall in the asset price at the time of a bust. This results in a pair of values $(\psi_L, \psi_H) = (1.94, 1.97)$. The other parameters remain the same as in Table 1.

We solve for the constrained planner's problem in the model with credit shocks and find that the behavior of the model economy is very similar to the case of output shocks. A planner would impose an optimal Pigouvian tax on borrowing of $\tau_H^{SS} = 0.61$ percent if the economy has reached its steady state during a boom. In a bust, the planner lowers the tax and slowly raises it back to its high steady-state value as the economy re-accumulates debt.

The general magnitude of the externality—and by implication of optimal policy measures targeted at internalizing it—therefore seems to depend not so much on the source of shocks as on the extent of amplification when the borrowing constraint becomes binding. The optimal policy measures in the economy are similar as long as we calibrate the model in a way that reproduces similar frequencies and magnitudes of crisis as our benchmark model with endowment shocks.

4.2 Debt Maturity

We have observed in the data that outstanding debt fell by substantially less than asset values in the US coporate and household sectors during the financial crisis. This implied a relatively low value of ϕ in the calibration. However, the small sensitivity of outstanding debt to collateral value could be due to the fact that a substantial fraction of the debt is medium- or long-term so that the full impact of low collateral values on outstanding debt is observed over several periods. We capture this idea in a tractable way by generalizing the collateral constraint (3) as follows,

$$\frac{w_{i,t+1} - (1-\delta)w_{i,t}}{R} + \delta(\psi + f(a_{i,t+1})p_t) \geq 0.$$

The parameter δ represents the fraction of the debt principal that comes due in any given period, i.e., the inverse of the duration of debt. The case of short-term debt corresponds to $\delta = 1$, which gives equation (3). In the general case $\delta < 1$, the collateral constraint applies to the flow of new debt issued in period t.

Using (4) with $a_i = 1$ and iterating backwards, the new constraint can also be written

$$\frac{w_{i,t+1}}{R} + \psi + \delta\phi \sum_{s=0}^{+\infty} (1-\delta)^s p_{t-s} \geq 0.$$

In this formulation, the collateral constraint has the same form as before, except that it involves the weighted average of past collateral prices. In a deterministic steady-state with constant asset price p_t, the constraint would simplify to our original collateral constraint (3). However, the dynamic behavior of the economy in case of shocks is modified: when the credit constraint binds, a unit decline in the current asset price reduces debt only by a fraction $\delta\phi$ as opposed to ϕ in our benchmark model. This mitigates the debt deflation dynamics in the economy.

Figure 9 illustrates how the optimal steady-state tax in the high state τ^H_{SS} varies with debt duration for the parameters of our benchmark calibration as listed in Table 1.[21] As we increase debt duration by moving leftwards in the graph from $\delta = 1$, the debt deflation effects that arise during binding constraints are mitigated. As a result, borrowers reduce their precautionary savings and the externality of a given dollar of debt at first rises. For $\delta \leq .78$, busts are sufficiently mild that a planner chooses not to insure against binding constraints when the steady state is reached. In this region, a planner uses macroprudential taxation only during the transition from a bust to the next boom in order to slow down the build-up of risk.[22]

[21] The derivation of the equilibrium with long-term debt and the numerical resolution method are presented in Appendices A.4 and B.2.

[22] Another effect of higher debt duration (lower δ) is to make the economy more resilient in the sense of admitting higher values of ϕ without leading to multiple equilibria—long-term debt insulates borrowers against self-fulfilling panics.

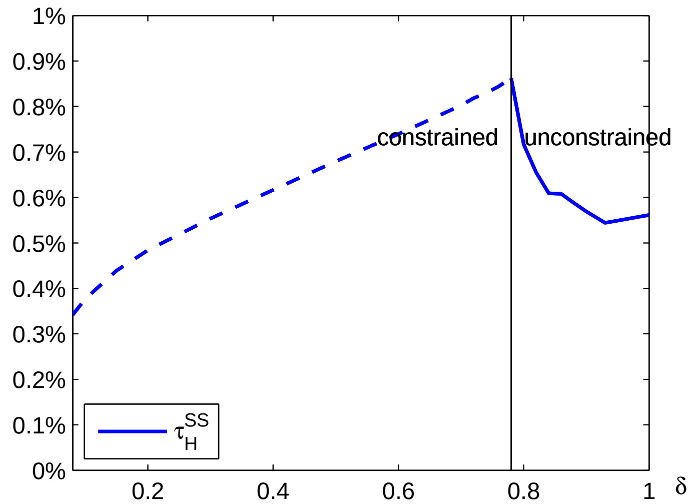

Figure 9: Dependence of macroprudential tax on inverse debt duration δ

4.3 Outside equity

We could assume that borrowers can sell equity rather than debt. Let us assume that the borrower can sell a claim on a share $s_t \leq \overline{s}$ of total income to outsiders. This claim will be sold to outsiders at price

$$\widetilde{p} = \frac{E(y)}{R-1},$$

and the budget constraint of borrowers becomes

$$c_{i,t} + a_{i,t+1}p_t + \frac{w_{i,t+1}}{R} = (1-\alpha)y_t + a_{i,t}\left(p_t + \alpha y_t\right) + w_{i,t} - s_{i,t}y_t + (s_{i,t+1} - s_{i,t})\widetilde{p}.$$

It is easy to see that the borrowers will always sell as much equity as possible, which is a way for them to insure against their income risk (at no cost since outsiders do not require a risk premium on equity) and benefit from the greater patience of outsiders. Thus, $s_t = \overline{s}$ in every period. The introduction of equity leads to a consumption boom, but the long-run equilibrium is the same as before except that total income is reduced by the factor $(1 - \overline{s})$. Our model is homogenous of degree 1 in income y, aside from the borrowing constraint. In a sample simulation in which we set $\overline{s} = .5$, we found the optimal macroprudential tax to be $\tau_H^{SS} = 0.59$, which is very close to the level in our benchmark model (0.56).

Although equity has better risk-sharing properties than debt, it is not used to reduce risk in the long run. Borrowers issue equity to increase their consumption, eventually leaving them with more liabilities and the same level of debt. Allowing borrowers to issue equity, thus, does not reduce the need for the prudential taxation of debt in the long run.

5 Conclusion

This paper has developed a simple model to study the optimal policy responses to booms and busts in credit and asset prices. We found that decentralized agents do not internalize that their borrowing choices in boom times render the economy more vulnerable to credit and asset price busts involving debt deflation in bust times. Therefore their borrowing imposes an externality on the economy. In our baseline calibration, a social planner would impose on average a relatively modest tax of half a percent per dollar on borrowing so as to reduce the debt burden and mitigate the decline in consumption in case of crisis.

The analysis presented in this paper could be extended in several directions. First, it would be interesting to analyze the case where the sensitivity of the credit constraint to the collateral price (parameter ϕ) is large enough to produce multiple equilibria and self-fulfilling asset price busts. This is the relevant case to consider if one wants to apply the model to leveraged financial institutions in systemic liquidity crises. The optimal Pigouvian tax is likely to be higher than with the calibrations that we have considered in this paper, but it is unclear whether the optimal tax should be binding in the long-run steady states. The optimal taxation might be implemented through the kind of countercyclical capital surcharges that are being discussed in the debates about the "macroprudential regulation" of banks. In addition, policies to remove the bad equilibria, such as lending-in-last-resort, may be appropriate.

Another direction of enquiry would take into account the effects of busts in asset prices and credit on production and income. Our model focused on the cost of excessive consumption volatility taking income as exogenous. In the real world, however, busts in credit and asset prices are likely to affect investment and other productive expenditures. It is not obvious a priori that investment will fall below the optimal level in a bust if it is triggered by a negative productivity shock, since in this case the demand for investment will fall at the same time as the credit constraint is tightened. The investment channel, however, might magnify the welfare cost of the fire sale externality in busts, and justify more aggressive prudential taxation in booms.

In addition, asset price and credit busts might have a permanent negative effect on long-run output. The data suggest that output does not generally catch up with its pre-crisis trend following a financial crisis (IMF, 2009). This will be the case, in our model, if the collateral constraint reduces productivity-enhancing expenditures. The welfare cost of asset price busts is likely to be larger in this case, leading to larger welfare gains from prudential taxation in booms, and a higher optimal Pigouvian tax level.

Finally, one would like to incorporate money to the model in order to derive insights for the debate on whether and how monetary policy should respond to credit and asset price booms. If there is nominal stickiness, a monetary restriction that raises the real interest rate in the boom should have the same macroprudential effect as the Pigouvian tax discussed in this paper. Such a preemptive restriction may come at a cost for the other objectives of monetary policy (e.g., in terms of inflation), and may not be necessary or desirable if the optimal Pigouvian taxation can be implemented independently of monetary policy. If monetary policy is the only available instrument, however, it stands to reason that it should be used with a prudential purpose in mind, at least at the margin.

6 References

Adrian, Tobias and Hyun Song Shin, 2010, "Liquidity and Leverage," *Journal of Financial Intermediation* 19(3), pp. 418-443.

Barro, Robert, 2009, "Rare Disasters, Asset Prices and Welfare Cost," *American Economic Review* 99(1), pp. 243-264.

Benigno, Gianluca, Huigang Chen, Christopher Otrok, Alessandro Rebucci and Eric R. Young, 2009, "Optimal Policy with Occasionally Binding Credit Constraints," manuscript, University of Virginia.

Benigno, Gianluca, Huigang Chen, Christopher Otrok, Alessandro Rebucci and Eric R. Young, 2010, "Financial Crises and Macro-Prudential Policies," manuscript, University of Virginia.

Benigno, Gianluca, Huigang Chen, Christopher Otrok, Alessandro Rebucci and Eric R. Young, 2011, "Revisiting Overborrowing and its Policy Implications," in Cespedes, Luis Felipe, Roberto Chang and Diego Saravia, *Monetary Policy under Financial Turbulence*, Central Bank of Chile, pp. 145-185.

Bernanke, Ben S., 2002, "Asset-Price 'Bubbles' and Monetary Policy," remarks before the New York Chapter of the National Association for Business Economics, New York, October 15, 2002.

Bernanke, Ben S. and Mark Gertler, 1989, "Agency Costs, Net Worth and Business Fluctuations," *American Economic Review* 79(1), 14-31.

Bernanke, Ben S., Mark Gertler, and Simon Gilchrist, 1999, "The Financial Accelerator in a Quantitative Business Cycle Framework," *Handbook of Macroeconomics*, Vol 1C, edited by John B. Taylor and Michael Woodford, Elsevier.

Bianchi, Javier, 2011, " Overborrowing and Systemic Crises in the Business Cycle," forthcoming, *American Economic Review.*

Bianchi, Javier, and Enrique Mendoza, 2010, "Overborrowing, Financial Crises, and 'Macroprudential' Taxes", manuscript, Department of Economics, University of Maryland.

Borio, Claudio, 2003, "Towards a Macroprudential Framework for Financial Supervision and Regulation?" *CESifo Economics Studies* 49(2), pp. 181-215.

Brunnermeier, Markus, 2009, "Deciphering the Liquidity and Credit Crunch," *Journal of Economic Perspectives* 23(1), 77-100.

Caballero, Ricardo J. and Arvind Krishnamurthy, 2003, "Excessive Dollar Debt: Financial Development and Underinsurance," *Journal of Finance* 58(2), 867-894.

Carroll, Christopher D., 2006, "The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems,' *Economics Letters*, 312-320.

Carroll, Christopher D., 2008, "Theoretical Foundations of Buffer Stock Saving," manuscript, Department of Economics, Johns Hopkins University.

Cordoba, Juan-Carlos and Marla Ripoll (2004), "Credit Cycles Redux," *International Economic Review* 45(4), pp. 1011-1046.

Greenwald, Bruce and Joseph Stiglitz, 1993, "Financial Market Imperfections and Business Cycles," *Quarterly Journal of Economics* 108(1), 77-114.

Gromb, Denis and Dimitri Vayanos, 2002, Equilibrium and welfare in markets with financially constrained arbitrageurs, *Journal of Financial Economics* 66(2-3), pp. 361-407.

IMF, 2009, "What's the Damage? Medium-Term Output Dynamics After Financial Crises," chapter 4, *World Economic Outlook*, 121-151.

Jeanne, Olivier and Anton Korinek, 2010a, "Excessive Volatility in Capital Flows: A Pigouvian Taxation Approach", *American Economic Review*, AEA Papers and Proceedings, 403-407.

Jeanne, Olivier and Anton Korinek, 2010b, "Managing Credit Booms and Busts: A Pigouvian Taxation Approach," *NBER Working Paper* 16377.

Jeanne, Olivier and Anton Korinek, 2012, "Macroprudential Regulation Versus Mopping Up After the Crash," manuscript, University of Maryland.

Jermann, Urban and Vincenzo Quadrini, 2010, "Macroeconomic Effects of Financial Shocks," manuscript, University of Southern California.

Kiyotaki, Nobuhiro and John H. Moore, 1997, "Credit Cycles," *Journal of Political Economy* 105(2), 211-248.

Kocherlakota, Narayana (2000), "Creating business cycles through credit constraints," *Federal Reserve Bank of Minneapolis Quarterly Review* 24, pp. 2-10.

Korinek, Anton, 2009, “Regulating Capital Flows to Emerging Markets: An Externality View,” manuscript, Department of Economics, University of Maryland.

Korinek, Anton, 2010, “Systemic Risk-Taking: Amplification Effects, Externalities, and Regulatory Responses,” manuscript, Department of Economics, University of Maryland.

Lorenzoni, Guido, 2008, "Inefficient Credit Booms," *Review of Economic Studies* 75(3), 809-833.

Mendoza, Enrique G., 2010, “Sudden Stops, Financial Crises and Leverage,” *American Economic Review* 100(5), pp. 1941-1966.

Mendoza, Enrique G. and Katherine Smith, 2006, "Quantitative Implications of a Debt-Deflation Theory of Sudden Stops and Asset Prices," *Journal of International Economics* 70, 82-114.

Nikolov, K., 2010, "Is Private Leverage Excessive?", MPRA Working Paper 28407.

Shleifer, Andrei and Robert Vishny, 2011, "Fire Sales in Finance and Macroeconomics," *Journal of Economic Perspectives* 25(1), 29–48.

Stein, Jeremy (2011), “Monetary Policy as Financial-Stability Regulation,” forthcoming, *Quarterly Journal of Economics.*

A Solution of Benchmark Model

A.1 Laissez-faire

Decentralized agents solve the Lagrangian

$$\mathcal{L}_t = E_t \Sigma_{s=t}^{+\infty} \beta^{t-s} \Big\{ u \Big((1-\alpha) y_s + a_{i,s}(\alpha y_s + p_s) + w_{i,s} - \frac{w_{i,s+1}}{R} - a_{i,s+1} p_s \Big) + \\ + \lambda_{i,s} \Big[\frac{w_{i,s+1}}{R} + \psi + \min(a_{i,s+1}, \phi) p_s \Big] \Big\}.$$

Given CRRA utility, and assuming $a_{i,s+1} > \phi$ (which must be true in a symmetric equilibrium), this implies the first-order conditions

$$\begin{aligned} FOC\,(w_{i,s+1}) \quad &: \quad c_{i,s}^{-\gamma} = \beta R E_s \left[c_{i,s+1}^{-\gamma} \right] + \lambda_{i,s}, \\ FOC\,(a_{i,s+1}) \quad &: \quad p_s c_{i,s}^{-\gamma} = \beta E_s \left[c_{i,s+1}^{-\gamma} (\alpha y_{s+1} + p_{s+1}) \right]. \end{aligned}$$

In a symmetric equilibrium with a representative agent, this gives (5) and (6).

A.2 Alternative Specifications of Borrowing Constraint

We checked the robustness of our results to alternative specifications of the constraint by numerically solving for the equilibrium and found that the qualitative and quantitative results are only marginally affected. First, one could assume that lenders can seize a fraction ϕ of the asset holdings (as e.g. in Bianchi and Mendoza, 2011). Then the collateral constraint (3) becomes

$$-\frac{w_{i,t+1}}{R} \leq \psi + \phi a_{i,t+1} p_t,$$

and the asset pricing equation (6) is replaced by

$$p_t = \frac{\beta E_t \left[u'(c_{i,t+1})(\alpha y_{t+1} + p_{t+1}) \right]}{u'(c_{i,t}) - \phi \lambda_{i,t}}.$$

The new term in the denominator, $\phi \lambda_{i,t}$, reflects the collateral value of the asset—i.e., the fact that borrowers take into account that greater asset holdings increase their borrowing capacity. The model's quantitative predictions, however, are virtually the same. For the parameters values of our benchmark calibration, the steady-state asset price is 1 percent higher under the alternative specification of the constraint, and the impact of busts on the asset price was 0.2 percent lower. The optimal macro-prudential tax in the boom steady state was 0.55 percent compared to 0.56 percent under the original specification of the constraint (3). Heuristically, the reason why these results differ only insignificantly is that ϕ is small in our calibrations in order to ensure a unique equilibrium, and that the marginal utilities $u'(c_{i,t})$ and $\beta RE \left[u'(c_{i,t+1}) \right]$ are relatively close to each other in the ergodic steady state of the economy.

Alternatively, if the borrowing limit depends on the holding of asset at the beginning of the period,

$$-\frac{w_{t+1}}{R} \leq \psi + \phi a_{i,t} p_t,$$

then the pricing equation becomes

$$p_t = \beta \frac{E_t \left[u'(c_{i,t+1})(\alpha y_{t+1} + p_{t+1}) + \phi \lambda_{i,t+1} p_{t+1} \right]}{u'(c_{i,t})}.$$

Again, we found that the quantitative results were virtually the same as with our model.

A.3 Social planner. Implementation with Pigouvian Taxes

If we solve the decentralized agent's problem under a tax that requires the agent to pay τ_t for every dollar borrowed as specified in budget constraint (18), then the first order condition is

$$FOC\left(w_{t+1}\right): \qquad \left(1-\tau_t\right) u'(c_t) = \lambda_t + \beta R E_t \left[u'(c_{t+1}) \right].$$

This condition replicates the first-order condition of the planner that is given above if the tax rate is set such that

$$\lambda_t + \beta R E_t \left[u'(c_{t+1}) + \phi \lambda_{t+1} p'(m_{t+1}) \right] = \lambda_t + \beta R E_t \left[u'(c_{t+1}) \right] + \tau_t u'\left(c_t\right).$$

Simplifying this expression yields the formula (19) given in the text.

A.4 Multiple Equilibria

This appendix gives a heuristic account of the mechanism underlying multiplicity and how assumption A.1 ensures uniqueness.[23] The multiplicity comes from the self-reinforcing loop that links consumption to the price of the collateral. In the constrained regime, a fall in the price of the collateral asset decreases the borrowers' level of consumption, which in turn tends to depress the price of the asset. This loop, which is essential for our results since it explains the financial amplification of real shocks, may also—if its effect is strong enough—lead to self-fulfilling crashes in the price of the asset.

More formally, the loop linking consumption to the asset price is captured by equations (6) and (8). The credit constraint (8) can be written

$$c \leq m + \psi + \phi \hat{p}(m, c), \tag{22}$$

where the function $\hat{p}(\cdot, \cdot)$ is given by equation (13).

For a given m, the right-hand side of (22) is increasing in c because the credit constraint on each individual is relaxed by a higher level of aggregate

[23] We are not aware of papers giving general conditions under which the equilibrium is unique in models of the type considered here (i.e., extensions of Carroll's (2008) analysis to the case with endogenous credit constraints).

consumption that raises the price of the asset.[24] Multiplicity may arise if the left-hand side and the right-hand side of (22) intersect for more than one level of c, which may occur if assumption A.1 is violated.

We explore the multiplicity of equilibria in the remainder of this section by considering a special case of the model that can be solved analytically – the case where y is constant and $\beta R = 1$ – and restricting our attention to multiplicity in period 1.

Assume that the economy starts from an initial level of wealth m_1 and let us solve by backward induction. A steady state equilibrium starting from period 2 in which the constraint is always loose is feasible if

$$m_2 \geq \bar{m} \equiv y - R(\psi + \phi p^{SS}).$$

where $\bar{m}$ is the threshold value of m_2 for which this steady state equilibrium is marginally unconstrained. In a steady state equilibrium, consumption and the asset price satisfy

$$c^{SS}(m_2) = \beta y + (1-\beta)m_2, \tag{23}$$

$$p^{SS} = \frac{\alpha y}{r}. \tag{24}$$

The equilibrium in period 1, given that the economy is in a steady state equilibrium in period 2, will satisfy

$$\begin{aligned} c_1 &= \min\left\{c^{SS}(m_2), m_1 + \psi + \phi p_1\right\} \\ p_1 &= \beta\left(\frac{c_1}{c_2}\right)^{\gamma}(\alpha y + p^{SS}) = p^{SS}\left(\frac{c_1}{c_2}\right)^{\gamma} \\ \text{where } m_2 &= y + R(m_1 - c_1) \end{aligned}$$

As captured by the first equation, the equilibrium in period 1 is either unconstrained or constrained. An unconstrained period 1 equilibrium is feasible if $m_1 \geq \bar{m}$. In that case, the economy jumps immediately to the steady state described by equations (23) and (24). A constrained equilibrium is defined by the binding constraint and can be determined as a solution to the implicit equation

$$c_1 = m_1 + \psi + \phi\hat{p}(m_1, c_1) \tag{25}$$

Using $c_2 = y + r(m_1 - c_1)$ to substitute out c_2, the price function in the constrained region can be written

$$\hat{p}(m_1, c_1) = p^{SS}\left(\frac{y + rm_1}{y + r(m_1 - c_1)} - 1\right)^{\gamma}. \tag{26}$$

If $\gamma \geq 1$ this is a strictly convex function of c_1 in the constrained region, as shown in Figure 10. Hence the slope of the r.h.s. of (22) reaches its maximum

[24] This is captured by the denominator $u'(c)$ on the r.h.s. of (13). However, because of the other terms in m', the sign of the variations of $\hat{p}$ with c is a priori ambiguous in our general model.

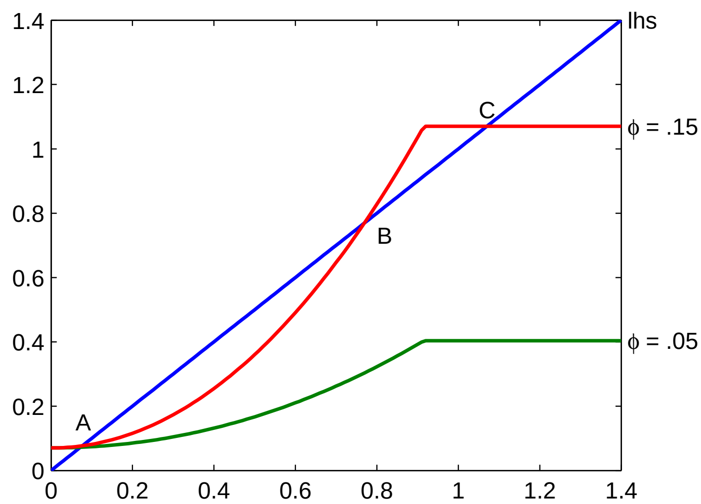

Figure 10: L.h.s. and r.h.s. of equation (22) for $\phi = .05$ and $\phi = .15$ (constant y and $\beta R = 1$)

when the economy enters the unconstrained region. This maximum is given by,

$$\phi \left. \frac{\partial \hat{p}}{\partial c_1} \right|_{c_1 = c^{SS}(m_1)} = \phi \left(1 + \frac{1}{r} \right) \frac{\gamma \alpha y}{c^{SS}(m_1)}. \tag{27}$$

As can be seen on Figure 10, equilibrium multiplicity is possible only if the unconstrained steady state equilibrium exists, which requires $m_1 \geq \overline{m}$. In addition, equilibrium multiplicity requires the slope of the r.h.s. of (22) to be larger than 1. To avoid multiplicity, it is sufficient that this slope be smaller than 1 at the kink for $m_1 = \overline{m}$, that is

$$\phi \left. \frac{\partial \hat{p}}{\partial c_1} \right|_{c_1 = c^{SS}(\overline{m})} < 1$$

$$\text{or} \qquad \phi \leq \hat{\phi} := \frac{y - r\psi}{\alpha y \left[\left(1 + \frac{1}{r} \right) \gamma + 1 \right]}.$$

If this condition is satisfied, then the slope of the r.h.s. of (22) is lower than 1 everywhere, so that assumption A.1 is satisfied and the equilibrium is unique. This is true not only for $m_1 = \overline{m}$ but also for any $m_1 \geq \overline{m}$ since the slope at the kink is decreasing in m_1 (see equation (27)). Conversely, if this condition is not satisfied, then there is multiplicity for m_1 slightly below $\overline{m}$.

If $m_1 < \bar{m}$, then the equilibrium is constrained and is given as the implicit solution to equation (25). The asset price satisfies $p_1 \leq p^{SS}$. Therefore we find that

$$m_2 = y - R\left(\psi + \phi p_1\right) \geq \bar{m}$$

and we validate our characterization (23) and (24) of the equilibrium in period 2 and onwards.

A.5 Long-Term Debt

The Lagrangian of our setup extended to long-term debt that repays a fraction δ every period is

$$\begin{aligned}\mathcal{L}_t &= E_t \Sigma_{s=t}^{+\infty} \beta^{t-s} \left\{ u \left((1-\alpha) y_s + a_{i,s}(\alpha y_s + p_s) + w_{i,s} - \frac{w_{i,s+1}}{R} - a_{i,s+1} p_s \right) + \right. \\ & \left. + \lambda_{i,s} \left[\frac{w_{i,s+1} - (1-\delta) w_{i,s}}{R} + \delta (\psi + \phi p_s) \right] \right\}.\end{aligned}$$

This changes the first-order condition on w_{s+1} to

$$FOC\,(w_{i,s+1}): \qquad u'(c_{i,s}) + \beta(1-\delta) E_s [\lambda_{i,s+1}] = \beta R E_s [u'(c_{i,s+1})] + \lambda_{i,s}.$$

Taking on more debt now not only has the benefit of raising current consumption, but also of having to roll over $(1-\delta)$ less debt next period, which is valuable if the borrowing constraint next period is binding. The remaining first-order conditions are unchanged.

When including long-term debt, we can no longer summarize the state variables in a single variable $m = w + y$, because w determines the level of debt that comes due in the current period independently of y. All policy functions are therefore functions of the pair of state variables (w, y).

The Euler equation of the planner who borrows in long-term debt is

$$FOC\,(w_{s+1}): \quad u'(c_s) + \beta(1-\delta) E_s [\lambda_{s+1}] = \beta R E_s \left[u'(c_{s+1}) + \delta \phi \lambda_{s+1} \frac{\partial \bar{p}}{\partial w}(w_{s+1}, y_{s+1}) \right] + \lambda_s.$$

B Numerical solution method

B.1 Benchmark model

We present the numerical method in the case where income is i.i.d. and binomially distributed ($y = y_H$ or y_L) but the method can easily be extended to the case where y follows a Markov process with more than two states.

We first define a grid $\mathbf{w}$ for wealth. The minimum value in the grid is $w_{\min} = -\psi - y_L$. In iteration step k, we start with a triplet of functions $c_k(m)$, $p_k(m)$ and $\lambda_k(m)$ where $c_k(m)$ and $p_k(m)$ are weakly increasing in m and $\lambda_k(m)$ is weakly decreasing in m. For each $w' \in \mathbf{w}$ we associate a quadruplet (c, p, λ, m) under the assumption that the equilibrium is unconstrained. We solve the system of optimality conditions from section 2.2 under the assumption that the borrowing constraint is loose, noting that $m' = y' + w'$:

$$\begin{aligned}
c^{\text{unc}}(w') &= \left\{\beta R E\left[c_k(m')^{-\gamma}\right]\right\}^{-\frac{1}{\gamma}}, \\
p^{\text{unc}}(w') &= \frac{\beta E\left\{c_k(m')^{-\gamma}\cdot\left[\alpha y' + p_k(m')\right]\right\}}{c^{\text{unc}}(w')^{-\gamma}}, \\
\lambda^{\text{unc}}(w') &= 0, \\
m^{\text{unc}}(w') &= c^{\text{unc}}(w') + \frac{w'}{R}.
\end{aligned}$$

In the same way, we can solve for the constrained branch of the system for each $w' \in \mathbf{w}$ s.t. $w'/R \leq -\psi$ under the assumption that the borrowing constraint is binding in the current period as

$$\begin{aligned}
p^{\text{con}}(w') &= \frac{-w'/R - \psi}{\phi}, \\
c^{\text{con}}(w') &= \left[\frac{\beta E\left\{c_k(m')^{-\gamma}\cdot\left[\alpha y' + p_k(m')\right]\right\}}{p^{\text{con}}(w')}\right]^{-\frac{1}{\gamma}}, \\
\lambda^{\text{con}}(w') &= c^{\text{con}}(w')^{-\gamma} - \beta R E\left[c_k(m')^{-\gamma}\right], \\
m^{\text{con}}(w') &= c^{\text{con}}(w') + \frac{w'}{R}.
\end{aligned}$$

We then determine the next period wealth threshold $\overline{w}$ such that the borrowing constraint is marginally binding in the unconstrained system, i.e., such that

$$\frac{\overline{w}}{R} + \psi + \phi p^{\text{unc}}(\overline{w}) = 0.$$

This is the lowest possible w' that the economy can support (any lower level would violate the collateral constraint). By construction of this threshold, $c^{\text{unc}}(\overline{w}) = c^{\text{con}}(\overline{w})$ for consumption as well as for the other policy variables. This threshold gives the level of m that marks the frontier between the unconstrained and the constrained regimes, $\overline{m} = m^{\text{unc}}(\overline{w}) = m^{\text{con}}(\overline{w})$. The lowest possible level of m is $\underline{m} = m^{\text{con}}(-R\psi) = -\psi$. One can check, using the equations above, that any $w' \in [\overline{w}, -R\psi]$ can be mapped into one unconstrained quadruplet $(c^{\text{unc}}(w'),\ p^{\text{unc}}(w'),\ \lambda^{\text{unc}}(w'),\ m^{\text{unc}}(w'))$ and one constrained quadruplet $(c^{\text{con}}(w'), p^{\text{con}}(w'), \lambda^{\text{con}}(w'), m^{\text{con}}(w'))$.

We can construct the step-$(k+1)$ policy function $c_{k+1}(m)$ for the interval $\underline{m} \leq m < \overline{m}$ by interpolating on the pairs $\{(c^{\text{con}}(w'), m^{\text{con}}(w'))\}_{w'\in\mathbf{w}}$ where $w' \in [\overline{w}, -R\psi]$, and then for the interval $m \geq \overline{m}$ by interpolating on the pairs $\{(c^{\text{unc}}(w'), m^{\text{unc}}(w'))\}_{w'\in\mathbf{w}}$ for $w' \geq \overline{w}$. The resulting consumption function $c_{k+1}(m)$ is again monotonically increasing in m. We proceed in the same manner for the policy functions $p_{k+1}(m)$ and $\lambda_{k+1}(m)$, which are, respectively, monotonically increasing and decreasing in m. The iteration process is continued until the distance between two successive functions $c_k(m)$ and $c_{k+1}(m)$ (or other policy functions) is sufficiently small.

The source code of the program is available at:
`http://www.korinek.com/download/boombust.m`

B.2 Model with Long-Term Debt

When including long-term debt, all policy functions are functions of the pair of state variables (w, y). We present the algorithm to solve for the laissez-faire equilibrium (the case with social planner is similar). We modify the procedure outlined above by adjusting one equation in the unconstrained solution

$$c^{\text{unc}}(w', y) = \left\{\beta R E\left[c_k(w', y')^{-\gamma} - \beta(1-\delta)\lambda_k(w', y')\right]\right\}^{-\frac{1}{\gamma}},$$

In the constrained solution, the following four equations have to be satisfied,

$$\begin{aligned}
p^{\text{con}}(w', y) &= -\frac{w' - (1-\delta)\, w^{\text{con}}(w', y)}{\delta\phi R} - \frac{\psi}{\phi}, \\
c^{\text{con}}(w', y) &= \left[\frac{\beta E\left\{c_k(w', y')^{-\gamma} \cdot \left[\alpha y' + p_k(w', y')\right]\right\}}{p^{\text{con}}(w', y)}\right]^{-\frac{1}{\gamma}}, \\
\lambda^{\text{con}}(w', y) &= c^{\text{con}}(w', y)^{-\gamma} + \beta(1-\delta)\, E\left[\lambda_k(w', y')\right] - \beta R E\left[c_k(w', y')^{-\gamma}\right], \\
m^{\text{con}}(w', y) &= c^{\text{con}}(w', y) + \frac{w'}{R} \quad \text{or} \quad w^{\text{con}}(w', y) = c^{\text{con}}(w', y) + \frac{w'}{R} - y.
\end{aligned}$$

This is a system of four equations with four unknowns: $p^{\text{con}}(w', y)$, $c^{\text{con}}(w', y)$, $\lambda^{\text{con}}(w', y)$ and $w^{\text{con}}(w', y)$. It can be solved numerically, and also analytically in the case $\gamma = 2$. We substitute $w^{\text{con}}(w', y)$ from the fourth equation into the first equation to obtain

$$p^{\text{con}}(w', y) = \frac{-\left\{w' \cdot \frac{R-1+\delta}{R} - (1-\delta)\left(c^{\text{con}}(w', y) - y\right)\right\} - \delta R\psi}{\phi\delta R}.$$

In combination with the second equation this yields

$$\left[c^{\text{con}}(w', y)\right]^{\gamma} + \frac{-(1-\delta)\left[c^{\text{con}}(w', y)\right] + w' \cdot \frac{R-1+\delta}{R} + (1-\delta)\, y + \delta R\psi}{\phi\delta\beta R E\left\{c_k(w', y')^{-\gamma} \cdot \left[\alpha y' + p_k(w', y')\right]\right\}} = 0.$$

For $\gamma = 2$, this is a quadratic equation that can be solved as

$$\begin{aligned}
c^{\text{con}}(w', y) = {} & \frac{1-\delta}{2\phi\delta\beta R E\left\{c_k(w', y')^{-\gamma} \cdot \left[\alpha y' + p_k(w', y')\right]\right\}} \pm \\
& \pm \sqrt{\left(\frac{1-\delta}{2\phi\delta\beta R E\left\{c_k(w', y')^{-\gamma} \cdot \left[\alpha y' + p_k(w', y')\right]\right\}}\right)^2 - \frac{w' \cdot \frac{R-1+\delta}{R} + (1-\delta)\, y + \delta R\psi}{\phi\delta\beta R E\left\{c_k(w', y')^{-\gamma} \cdot \left[\alpha y' + p_k(w', y')\right]\right\}}}.
\end{aligned}$$

The equation has a solution if the discriminant is non-negative, or

$$(1-\delta)^2 \geq \phi\delta\beta R\left[w' \cdot \frac{R-1+\delta}{R} + (1-\delta)\, y + \delta R\psi\right] \cdot E\left\{c_k(w', y')^{-\gamma} \cdot \left[\alpha y' + p_k(w', y')\right]\right\}.$$

Note that for $\delta = 1$, the solution to the quadratic equation just reduces to the earlier condition

$$c^{\text{con}}(w', y) = \sqrt{-\frac{w'/R + \psi}{\phi\beta E\left\{c_k(w', y')^{-\gamma} \cdot \left[\alpha y' + p_k(w', y')\right]\right\}}}.$$