A Simple Test of Completeness in a Class of Nonparametric Specification*

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Abstract

This paper provides a test for completeness in a class of nonparametric specification with an additive and independent error term. It is known that such a nonparametric location family of functions is complete if and only if the characteristic function of the error term has no zeros on the real line. Because a zero of the error characteristic function implies that of an observed marginal distribution, we propose a simple test for zeros of characteristic function of the observed distribution, in which rejection of the null hypothesis implies the completeness. This test is applicable to many popular setting, such as nonparametric regression models with instrumental variables, and nonclassical measurement error models. We describe the asymptotic behavior of the tests under the null and alternative hypotheses and investigate the finite sample properties of the proposed test through a Monte Carlo study. We illustrate our method empirically by estimating a measurement error model using the CPS/SSR 1978 exact match file.

Keywords: *Completeness test, Nonparametric IV regression models, Nonclassical measurement error models.*

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1. Introduction

In this paper,we consider testability of completeness in a nonparametric class. The completeness conditions have been employed in the nonparametric identifications of many econometric models including nonparametric IV regression models, nonclassical measurement error models, and panel data models, etc. The completeness condition can be expressed in terms of a family of functions as follows: For all measurable real functions m such that $E[|m(X)|] < \infty$, and

$$\int m(x)f(x,z)dx = 0 \text{ a.e. in } \mathcal{Z}, \tag{1}$$

then $m(\cdot) = 0$ a.e.. Bounded completeness is similarly defined by stating that the only solution to Eq. (1) among all bounded functions is $m(\cdot) = 0$ a.e.. In this paper, we focus on testing issues on bounded completeness and refer the family $\{f(x,z) : z \in \mathcal{Z}\}$ satisfying the above restriction as a complete family. Define the set of all absolutely integrable and bounded functions with domain $\mathscr{A}$ as $L^1_{bnd}(\mathscr{A}) = \{h(\cdot) : \int_{\mathcal{X}} |h(x)|^1 dx < \infty$ and $\sup_{a\in\mathscr{A}} h(a) < \infty\}$, where $\mathscr{A}$ is a closed interval in $\mathbb{R}$. We can rewrite Eq. (1) as an integral operator with the kernel function $f(x,z)$ through the following:

$$(L_f h)(z) = \int_{\mathcal{X}_z} h(x) f(x,z)dx, \tag{2}$$

where L_f is an integral operator from $L^1_{bnd}(\mathcal{X})$ to $L^1_{bnd}(\mathcal{Z})$. The completeness of the family $\{f(x,z) : z \in \mathcal{Z}\}$ over $L^1_{bnd}(\mathcal{X})$ is equivalent to the injective property of the integral operator L_f using $f(x,z)$ as a kernel function.

The injectivity of the conditional expectation operator using the conditional distribution $f(X|Z)$ as a kernel function is used to obtain the nonparametric identification of nonparametric IV regression models (see Newey and Powell (2003); Ai and Chen (2003); Chernozhukov and Hansen (2005); Blundell, Chen, and Kristensen (2007); Chernozhukov, Imbens, and Newey (2007); Horowitz and Lee (2007); Darolles, Fan, Florens, and Renault (2011); Horowitz (2011)). Hu, Schennach, and Shiu (2017) use the result of the Volterra equation to provide sufficient conditions for nonparametric identification of IV regression models in compact supports. Further, as discussed in Horowitz (2012) an identification condition may not exist when an instrument is not valid. This raises the question whether it is possible to test for the completeness. Canay, Santos, and Shaikh (2013) consider the hypothesis testing problems for testing completeness in the nonparametric IV regression model and show the completeness condition is, without further restrictions, untestable against very general alternatives. Any test that controls asymptotic size will have trivial asymptotic power against any alternative because distributions for which completeness fails are arbitrarily close to distributions for which completeness holds. Freyberger (2017) provides a test for a restricted completeness by linking the outcome of the test to consistency of an estimator.

The method developed in this paper builds on Theorem 2.1 in Mattner (1993), the nonparametric location family of functions $\{f_V(x-z) : z \in \mathbb{R}\}$ is complete if and only if the characteristic function of V is everywhere nonvanishing. Under nonparametric specifications for an additive functional form and an independent error term, the everywhere nonvanishing property of the characteristic function of observables is a sufficient condition for the completeness condition. This enables us to construct test statistics for the completeness using the squared modulus of empirical characteristic functions. Compared with the other tests for completeness, the test statistics are relatively simple because they are based on marginal distributions of observables instead of joint distributions.[1] One of the advantage of the property is that the test statistics can be used to test completeness conditions related to unobservables. Under the nonparametric specifications, rejection of the null hypothesis implies the nonparametric family of conditional density functions $\left\{f_V(x-z) : z \in \mathbb{R}\right\}$ is complete in $L^1_{bnd}(\mathscr{X})$. Our nonparametric restrictions on the class of functions are strong enough to allow testability for completeness. We illustrate the propose simple test statistics for the completeness conditions in nonparametric IV regression models, and nonclassical measurement error models with instrumental variables.

The completeness condition has been used to obtain global or local identification in a variety of nonparametric econometric models other than nonparametric IV models such as measurement error models (see Hu and Schennach (2008); Carroll, Chen, and Hu (2010); Chen and Hu (2006)), and panel data models (see Shiu and Hu (2013)), etc.. Several papers including Newey and Powell (2003), Andrews (2017), D'Haultfoeuille (2011), and Hu and Shiu (2018) have provided sufficient conditions for different versions of completeness.

There are three major implications of the results in our paper:

1. Uniform or point-wise tests

This paper provides a useful result for the test of completeness condition in a class of models based on convolution. This result is complementary to the non-testability result of the completeness condition in Canay, Santos, and Shaikh (2013). They consider a very general class of

[1] Other tests for completeness such as a full rank test for completeness of discrete cases in Robin and Smith (2000), and a test for a restricted version of completeness in Freyberger (2017) are derived in terms of joint distributions.

models and show that any test that controls asymptotic size uniformly over a large class of non-complete distributions has trivial asymptotic power against any alternative. Denote $\mathbf{P}_0$ as some class of distributions where the completeness fails and $\mathbf{P}_1$ is the class of distributions where the completeness holds. A uniform result on size control is to control size uniformly over $\mathbf{P}_0$. Canay, Santos, and Shaikh (2013) show any distribution in $\mathbf{P}_1$ can be arbitrarily approximated by a sequence of distributions in $\mathbf{P}_0$. This means the impossibility of having a nontrivial test that controls size over a large set of possible DGP's. Within the class of models considered in our paper, $\mathbf{P}_0$ corresponds to the set of vanishing ch.f., i.e., characteristic functions with zeros on the real line, and $\mathbf{P}_1$ corresponds to the set of non-vanishing ch.f.. It can be shown that none of non-vanishing ch.f. can be arbitrarily approximated by a sequence of vanishing ch.f.. That is where the testability of completeness comes from in our paper.

On one hand, a general result on testability is of interest in econometric theory; On the other hand, practitioners usually work on a specific model in empirical applications and want to know what can be tested for such a specific model instead of an extremely general model. Since our test only forcuses on a class of models, it would control size point-wisely for any distribution satisfying the null hypothesis but would not control size uniformly. If an empirical model falls into our class of specifications, this paper shows that testing the completeness condition is feasible and actually simple. In that sense, our results are complementary to the result in Canay, Santos, and Shaikh (2013). Therefore, we believe the point-wise results in our paper are very useful for empirical research using this class of models, especially given the existing uniform non-testibility result.

2. Non-testability of continuity

The same uniformity argument can be applied to tests of the continuity assumption. Suppose we consider a general nonparametric regression model $Y = m(X) + \eta$ with $X \in \mathbb{R}$ and want to test the continuity assumption imposed on the regression function $m(\cdot)$ over the real line using a random sample of $\{Y, X\}$. Without imposing enough restrictions, one can establish a non-testability result of the continuity assumption simply because we only observe a countable number of possible values in the support of regressor X as the sample size goes to infinity. One can always find a discontinuous function which is observationally equivalent to the true continuous regression function $m(\cdot)$. In other words, a continuous function over the real line only exists at the population level. Such a non-testability result is empirically vacuous.

Furthermore, the results in Canay, Santos, and Shaikh (2013) actually rely on such a continuity restriction. In their paper, the completeness of $f(X|Z)$ is defined on the whole support of two continuous variables X and Z with $X \in \mathbb{R}$ and $Z \in \mathbb{R}$. They show that one can always use a sequence of discontinuous step functions to approximate the continuous distribution function $f(X|Z)$ defined on the support. The completeness, which is defined in a functional space of continuous functions over $\mathbb{R}$, doesn't hold with these discontinuous step functions. [2] Given a random sample, the limit of that sequence of discontinuous step functions is observationally equivalent to the true continuous distribution function $f(X|Z)$. That is why the completeness condition is not testable in this general setting. In other words, their proof of the non-testability of completeness is in fact based on the non-testability of the continuity restriction. That is also why they directly rule out cases with a discrete X. The full support of a discrete X can be identified in a large sample. It is well known that completeness is the same as the full rank condition of a matrix, which is testable, in the discrete case.

The key of the testability of the completeness condition actually lies in the case where X has a support with infinitely countable points. Because one can only observe or identify such a support with a random sample as the sample size goes to infinity, restrictions imposed beyond such a support will have to be put into non-testable assumptions. Whether completeness is testable in this countable discrete case is an open question for future research.

3. Bounded completeness – identification of density functions

Our paper tests bounded completeness, i.e., completeness over a space of bounded functions. One argument against considering the set of bounded function is that it rules out polynomials, in particular, linear functions over the real line. In a standard nonparametric IV model, we need the completeness of $f(X|Z)$ to identify regression function $m(\cdot)$ from $E[Y|Z] = \int m(X)f(X|Z)dX$. Therefore, bounded completeness is not enough to nonparametrically identify $m(\cdot)$ in the case where the support of X is the whole real line and $m(\cdot)$ is linear. However, the bounded completeness is very useful in measurement error models, where the goal is to identify the density function of a latent variable X^* from $f(X) = \int f(X|X^*)f(X^*)dX^*$. The key identification assumption is the completeness of $f(X|X^*)$. In this case, bounded completeness is useful enough even if the support of X^* is the whole real line because it is a quite mild restriction to assume $f(X^*)$ is bounded. Our tests are based on a convolution setting, where $f(X|X^*) = f_\epsilon(X - X^*)$ with a classical measurement error ϵ. In such a convolution setting, completeness has a simple implication, i.e., a non-vanishing ch.f. of ϵ. Under the so-called non-differential measuremen-

[2] It is possible that a sequence of step functions is complete in the functional space of continuous functions over a compact set. For example, a sequence of the so-called Haar wavelet functions, which are discontinuous step functions, can uniformly approximates any continuous real function with compact support.

t error assumption, we may consider the relationship between dependent variable Y and X^* though $f(X,Y) = \int f(X|X^*)f(X^*,Y)dX^*$. Notice that bounded completeness is enough to identify the joint density $f(X^*,Y)$. That means we can also identify the conditional mean function $E[Y|X^*] = \int Yf(Y|X^*)dY$, which doesn't need to be bounded even if the support of Y is the whole real line. In that sense, the possible unboundedness of the mean function is due to the unboundedness of the function, i.e., g(Y)=Y, of which we are taking expectation $E[g(Y)|X^*]$, while the density function $f(Y|X^*)$ is usually bounded. In other words, bounded completeness is still useful for models with an unbounded conditional mean function through the identification of the corresponding density function.

The rest of this paper is organized as follows. Section 2 gives sufficient conditions for existence of a complete nonparametric family $\{f(x,z) : z \in \mathcal{Z}\}$ in $L^1_{bnd}(\mathcal{X})$. Section 3 describes the several asymptotic properties related to the squared modulus of the empirical characteristic functions. Section 4 applies the asymptotic results in Section 3 to construct test statistics for nonparametric IV regression models, and nonclassical measurement error models with instrumental variables. Then, we show the asymptotic behavior of the test under the null and alternative hypotheses. Section 5 provides the Monte Carlo study. In Section 6, we apply our test statistic in an empirical study using the CPS/SSR 1978 exact match file. Section 7 concludes. All technical proofs are in the Appendix.

2. Sufficient Conditions for Completeness

Although the non-testable result in Canay, Santos, and Shaikh (2013) has been established in a very general settings, we can provide a test for the completeness for a subclass of conditional density functions. The next lemma is a directly from Theorem 2.1 in Mattner (1993).

Lemma 2.1. *The nonparametric family $\{f_V(x-z) : z \in \mathbb{R}\}$ is complete in $L^1_{bnd}(\mathcal{X})$ if and only if the characteristic function of V is everywhere nonvanishing.*

A range of a function m is denoted by Range$(m) = \{m_1 : m_1 = m(z_1)$ for some $z_1 \in \mathcal{Z}\}$, where $\mathcal{Z}$ is the support of z. We can write the result as follows.

Lemma 2.2. *Suppose Range$(m) = \mathbb{R}$ and V is independent of Z. Consider*

$$X = m(Z) + V. \tag{3}$$

Then, the nonparametric family of conditional density functions $\left\{f(x|z) = f_V(x - m(z)) : z \in \mathcal{Z}\right\}$ *is complete in* $L^1_{bnd}(\mathcal{X})$ *if and only if the characteristic function of* V *is everywhere nonvanishing.*

Let $i = \sqrt{-1}$ be the unit imaginary number. Define the marginal characteristic functions ϕ_X, ϕ_m and ϕ_V by $\phi_X(t) = \mathrm{E}\left[e^{itX}\right]$, $\phi_m(t) = \mathrm{E}\left[e^{itm(Z)}\right]$ and $\phi_V(t) = \mathrm{E}\left[e^{itV}\right]$, respectively. Given V is independent of Z and $X = m(Z) + V$, we have

$$\phi_X(t) = \mathrm{E}\left[e^{itX}\right] = \mathrm{E}\left[e^{itm(Z)}\right] \cdot \mathrm{E}\left[e^{itV}\right] = \phi_m(t) \cdot \phi_V(t). \tag{4}$$

This implies that nonzero points of the characteristic function of X are also nonzero points of the characteristic function of V. If the characteristic function of X is everywhere nonvanishing, then the characteristic function of V is also everywhere nonvanishing.

Proposition 2.1. *Consider* $X = m(Z) + V$, *where* $Range(m) = \mathbb{R}$ *and* V *is independent of* Z. *If the characteristic function of* X *is everywhere nonvanishing, then the nonparametric family of conditional density functions* $\left\{f(x|z) = f_V(x - m(z)) : z \in \mathcal{Z}\right\}$ *is complete in* $L^1_{bnd}(\mathcal{X})$.

Under the range restriction and the independent condition, the characteristic functions $\phi_X(t)$ do not vanish on the real line is a sufficient condition for the completeness which is testable. The common distribution families such as the normal, chi-squared, Cauchy, gamma, Student, Laplace, and α-stable and exponential distributions have this non-vanishing property for their characteristic functions.

D'Haultfoeuille (2011) extends the nonparametric additive models with independent errors in Eq. (3) to the following nonparametric models with an additive separability:

$$X = \Lambda(m(Z) + V). \tag{5}$$

We summarize part (i) of Theorem 2.1 in D'Haultfoeuille (2011) in the following lemma.

Lemma 2.3. *Suppose Eq. (5) hold. Assume* $Range(m) = \mathbb{R}$ *and* V *is independent of* Z. *If the characteristic function of* V *is smooth or equivalently, of class* C^∞ *and everywhere nonvanishing, then the nonparametric family* $\{f(x|z) : z \in \mathbb{R}\}$ *is complete in* $L^1_{bnd}(\mathcal{X})$.

Thus, under the nonparametric specifications in Eqs. (3) and (5), the completeness condition is more accessible and the high level completeness conditions required for identifications in many econometric models can be verify in practice by examining the nonvanishing property of characteristic functions.

3. Asymptotic Properties of the Squared Modulus of Empirical Characteristic Functions

In this section, we will provide large sample results of the squared modulus of empirical characteristic functions and then use the result to construct test statistics in the next section. The empirical characteristic function is defined as

$$\phi_{X,n}(t) = \frac{1}{n}\sum_{j=1}^{n} e^{itX_j} \tag{6}$$

where $X_i, i = 1,...,n$ is an i.i.d. sequence of random variables. The empirical characteristic function is directly calculated from the empirical distribution and all the calculation is done in the complex domain. Because the characteristic function has a one-to-one correspondence with the distribution function, the empirical characteristic function retains all the information present in the sample. The asymptotic theory for the empirical characteristic function in the i.i.d. case is well understood in Feuerverger and Mureika (1977). Because the sufficient conditions for completeness is related to the nonvanishing property of characteristic function, we consider the squared modulus of the characteristic function and the empirical characteristic function and denote them as

$$a_X(t) = |\phi_X(t)|^2, \text{ and } a_{X,n}(t) = |\phi_{X,n}(t)|^2. \tag{7}$$

The squared modulus of the empirical characteristic function can be reduced to the following expression

$$a_{X,n}(t) = \frac{1}{n} + \frac{2}{n^2}\sum_{1\le j<k\le n} \cos(t(X_j - X_k)). \tag{8}$$

The following two results are straightforward consequence by Feuerverger and Mureika (1977) and Murota and Takeuchi (1981) and they are adopted to show the asymptotic properties of convergence of the squared modulus of the empirical characteristic function to squared modulus of the true characteristic function.[3]

[3] Related results can be found as Theorem 2.1 in Feuerverger and Mureika (1977) and Theorem 1 in Murota and Takeuchi (1981).

Proposition 3.1. *(Strong Law of Large Numbers) For fixed $T < \infty$,*

$$P\left(\lim_{n\to\infty} \sup_{|t|<T} |a_{X,n}(t) - a_X(t)| = 0\right) = 1. \tag{9}$$

Proposition 3.2. *(Asymptotic Normality) For fixed $T < \infty$. Let $R_n(t)$ be a stochastic process that is a residual of the squared modulus of the empirical characteristic function and the squared modulus of the population characteristic function:*

$$R_n(t) = \sqrt{n}\left(a_{X,n}(t) - a_X(t)\right),\ \textit{for } t \in [-T, T]. \tag{10}$$

As $n \to \infty$, the random process $R_n(t)$ converges to a zero mean Gaussian process $R(t)$ satisfying $R(t) = R(-t)$ and

$$E\Big[R(t)R(s)\Big] = 2Re\{\phi_X(-t)\phi_X(-s)\phi_X(t+s) + \phi_X(-t)\phi_X(s)\phi_X(t-s)\} - 4a_X(t)a_X(s). \tag{11}$$

Since Proposition 3.2 provides the asymptotic behavior of the squared modulus of the empirical characteristic function for a fixed single point t, any statistical procedure developed based on the behaviour may depend on the choice of the value of t. If a characteristic function fails the nonvanishing property, then there exists at least one zero point of the characteristic function. Information of the location of this zero point is essential for the inferential procedures based the asymptotic behavior of the squared modulus of the empirical characteristic function. Therefore, a potential test for the completeness is the problem of estimating the first zero point of the squared modulus of a characteristic function as well as an application of the inferential procedures to the estimated zero point.

If $a_X(t_0) = 0$ for some t_0, then $a_X(-t_0) = 0$. Thus, by the symmetry, we only consider $a_X(t)$ on the positive half line. Set

$$A_0 = \inf\{t > 0 : a_X(t) = 0\}, \tag{12}$$

and if A_0 does not exist, we denote $A_0 = \infty$, and define the random variable

$$A_n = \inf\{t > 0 : a_{X,n}(t) = 0\}. \tag{13}$$

We first show the convergence result of A_n to A_0 and then present some results concerning the estimation of A_n.

Theorem 3.1. *Suppose $A_0 < \infty$ is an isolated zero of $a_X(t)$, and $a_X(t)$ is smooth in some neighborhood of A_0. Then,*

$$A_n \xrightarrow{a.s.} A_0 \text{ as } n \to \infty. \tag{14}$$

From Eqs. (52) and (53), the locations of minimums are determined by the derivatives $\frac{\partial a_X(t)}{\partial t}$ and $\frac{\partial a_{X,n}(t)}{\partial t}$. Using Eq. (8), we have the following U statistic expression for $\frac{\partial a_{X,n}(t)}{\partial t}$

$$\frac{\partial a_{X,n}(t)}{\partial t} = -\frac{2}{n^2} \sum_{1 \le j < k \le n} \sin(t(X_j - X_k))(X_j - X_k). \tag{15}$$

Applying asymptotic normality result of U statistics[4], we obtain

Proposition 3.3. *For a fixed $T < \infty$, if $E(|X|) < \infty$, for $t \in [-T, T]$, we have*

$$\sqrt{n}\left(\frac{\partial a_{X,n}(t)}{\partial t} - \frac{\partial a_X(t)}{\partial t}\right) \xrightarrow{d} N(0, \sigma^2(t)), \tag{16}$$

where $\sigma^2(t) < \infty$.

Following the approach in Heathcote and Hüsler (1990), we can prove the limiting behavior of $\sqrt{n}(A_n - A_0)$.

Theorem 3.2. *Suppose $E(|X|) < \infty$, $A_0 < \infty$ is an isolated zero of $a_X(t)$, and $a_X(t)$ is smooth in some neighborhood of A_0. Set $a_X''(A_0) = \frac{\partial^2 a_X(t)}{\partial^2 t}|_{t=A_0}$ and $\sigma(t)$ is the asymptotic variance in Proposition 3.3. For $r_n = A_0 + \frac{z\sigma(A_0)}{\sqrt{n} a_X''(A_0)}$ with $z \in \mathbb{R}$, as $n \to \infty$, we have*

$$P\{A_n \le r_n\} \to \Phi(z) \text{ for every } z \in \mathbb{R}, \tag{17}$$

where Φ is the CDF of the standard normal distribution. In other words,

$$\sqrt{n}(A_n - A_0) \xrightarrow{d} N(0, \frac{\sigma(A_0)^2}{a_X''(A_0)^2}) \tag{18}$$

For $n > 1$, it is infeasible to calculate A_n explicitly, because the equation $a_{X,n}(t) = 0$ often does not have a unique root and standard approximation methods may fail. Welsh (1986) presents a simple explicit method of calculation of the first positive zero of the real part of a characteristic

[4]See Chapter 5 of Serfling (2009) for detailed results.

function which requires only a fractional moment condition on the distribution of X. We will follow the approach in Welsh (1986) to develop an iterative procedure for calculating a realisation of A_n, and establish almost sure convergence to A_0.

Let $s \in [0, A_n)$. Then, for any $t \in (s, A_n)$, using the expression in Eq. (8), we have

$$\begin{aligned}
|a_{X,n}(t) - a_{X,n}(s)| <= &\frac{2}{n^2} \sum_{1\le j<k\le n} \left| \cos(t(X_j - X_k)) - \cos(s(X_j - X_k)) \right| \\
= &\frac{2}{n^2} \sum_{1\le j<k\le n} \left| \sin\left(\frac{(t-s)(X_j - X_k)}{2} \right) \right| \\
\le &2^{1-\alpha} |t-s|^{\alpha} m_{\alpha}, \quad 0 < \alpha \le 1,
\end{aligned} \tag{19}$$

where

$$m_{\alpha} = \frac{1}{n^2} \sum_{1\le j<k\le n} \left| X_j - X_k \right|^{\alpha}, \quad 0 < \alpha \le 1. \tag{20}$$

This implies that for $t \in (s, A_n)$,

$$a_{X,n}(s) - 2^{1-\alpha} |t-s|^{\alpha} m_{\alpha} \le a_{X,n}(t). \tag{21}$$

The left hand side of the above inequality is an approximation of $a_{X,n}(t)$ on (s, A_n) and its zero is an approximation of A_n. Set $T_{X,n,0} = 0$ and

$$T_{X,n,k+1} = T_{X,n,k} + \left(\frac{a_{X,n}(T_{X,n,k})}{2^{1-\alpha} m_{\alpha}} \right)^{\frac{1}{\alpha}}, \quad k = 0, 1, 2, \ldots \tag{22}$$

The notation F_x refers to the CDF of X. The asymptotic properties of $T_{X,n,k}$ are summarized in the following theorem.

Theorem 3.3. *(i) For each fixed $n < \infty$, $T_{X,n,k}$ is a monotone increasing sequence which converges to A_n, almost surely as $k \to \infty$; (ii) If $A_0 < \infty$ is an isolated zero of $a_X(t)$, $a_X(t)$ is smooth in some neighborhood of A_0, and*

$$\int\int |x_1 - x_2|^{\alpha} dF_{x_1} dF_{x_2} < \infty \tag{23}$$

for some $0 < \alpha \le 1$, then for N large enough

$$\sup_{n\ge N} |T_{X,n,k} - A_n| \xrightarrow{a.s.} 0 \text{ as } k \to \infty. \tag{24}$$

(iii)Suppose the assumptions in (ii) hold. Then, for a sufficiently large n, there exists k depending on n such that

$$|T_{X,n,k} - A_n| \le \frac{1}{\sqrt{n}} \; a.s. \; . \tag{25}$$

4. Testing Completeness in Nonparametric Specifications

In this section, we will apply the asymptotic results of the squared modulus of the empirical characteristic functions in Section 3 to provide simple tests for the completeness that is used for identifying several econometric models. We will construct test statistics for nonparametric IV regression models with or without a convolution structure between x and z, and nonclassical measurement error models with instrumental variables. Because the structures of the two models are different, we discuss the testing issues in the context of these models.

4.1. Nonparametric IV Regression Models

Consider a nonparametric IV regression model as follows:

$$y = \mu(x) + v, \text{ with } \mathrm{E}[v|z] = 0. \tag{26}$$

We observe a random sample of $\{Y, X, Z\}$, and denote the supports of these random variables as $\mathscr{Y}$, $\mathscr{X}$ and $\mathscr{Z}$, respectively. The conditional expectation of Eq. (26) has the following integral equation

$$E[y|z] = \int \mu(x) f(x|z) dx \text{ a.e. in } \mathscr{Z}. \tag{27}$$

The object of interest is the unknown function $\mu(\cdot)$ which is not observable from the distribution of $\{Y, X, Z\}$. For the identification of the function $\mu(\cdot)$, Newey and Powell (2003) and Darolles, Florens, and Renault (2006) imposed the completeness condition for the conditional distribution $f(X|Z)$. Suppose there exists μ_1 and μ_2 satisfy

$$E[y|z] = \int \mu_1(x) f(x|z) dx = \int \mu_2(x) f(x|z) dx. \tag{28}$$

This implies that

$$0 = \int (\mu_1(x) - \mu_2(x)) f(x|z) dx \text{ a.e. in } \mathcal{Z}. \tag{29}$$

The completeness of the nonparametric family of conditional density functions $\left\{ f(x|z) : z \in \mathcal{Z} \right\}$ ensures that there is a unique solution $\mu = \mu_1 = \mu_2$ in $L^1_{bnd}(\mathcal{X})$.

Assumption 4.1. *(i) the range of the conditional mean function $E[X|Z = z]$ is the whole real line;(ii) write $X = E[X|Z = z] + V$, and assume V is independent of Z.*

Proposition 2.1 implies that under Assumption 4.1, if the characteristic function of X is everywhere nonvanishing, then the nonparametric family of conditional density functions $\left\{ f(x|z) = f_V(x - \mathrm{E}[X|Z = z]) : z \in \mathcal{Z} \right\}$ is complete in $L^1_{bnd}(\mathcal{X})$. Therefore, a test for an everywhere nonvanishing characteristic function of X under Assumption 4.1 can be regarded as a test for completeness.

The null hypothesis of the test is

$$H_0 : \phi_X(t) = 0 \text{ for some } t. \tag{30}$$

The alternative hypothesis is

$$H_1 : \phi_X(t) \neq 0 \text{ for all } t. \tag{31}$$

As explained in Proposition 2.1, under Assumption 4.1, H_1 is true implies the completeness of $\left\{ f(x|z) = f_V(x - \mathrm{E}[X|Z = z]) : z \in \mathcal{Z} \right\}$ in $L^1_{bnd}(\mathcal{X})$. Under H_0, we have the A_0 in Eq. (12) exists and finite and $a_X(A_0) = 0$. We may want to apply the asymptotic result of Proposition 3.2 at the point A_0 to construct a test statistic. Theorem 3.3 ensures the iterative estimator $T_{X,n,k}$ in Eq. (22) satisfying $T_{X,n,k} \xrightarrow{a.s.} A_0$ as $n, k \to \infty$. Given the estimator $T_{X,n,k}$ for A_0, we apply Proposition 3.2 at the estimator $T_{X,n,k}$ to obtain

$$R_n(T_{X,n,k}) = \sqrt{n} \left(a_{X,n}(T_{X,n,k}) - a_X(T_{X,n,k}) \right) \tag{32}$$

converges to a zero mean Gaussian process with the variance $\mathrm{E}\left[R(T_{X,n,k})^2 \right]$ as $n \to \infty$. Under H_0, the term $a_X(T_{X,n,k})$ in Eq. (32) is close to zero as $a_X(T_{X,n,k}) \xrightarrow{a.s.} a_X(A_0) = 0$ as $n, k \to \infty$. On the other hand, under H_1, the term $a_X(T_{X,n,k})$ is nonzero. According to the results, we construct a test statistic with

$$\tau_{Xn} = \sqrt{n} a_{X,n}(T_{X,n,k}). \tag{33}$$

The assumptions in Theorem 3.2 and Theorem 3.3 are needed to derive the asymptotic properties of τ_{Xn} and we collect them in the next assumption.

Assumption 4.2. *Denote* $A_{X0} = \inf\{t > 0 : a_X(t) = 0\}$. *(i) Suppose* $E(|X|) < \infty$, A_{X0} *is an isolated zero of* $a_X(t)$, *and* $a_X(t)$ *is smooth in some neighborhood of* A_{X0};*(ii)*

$$\int\int |x_1 - x_2|^{\alpha} dF_{x_1} dF_{x_2} < \infty. \tag{34}$$

Since we have known the sampling distribution of τ_{Xn} asymptotically converges to a zero mean Gaussian process with the variance $\mathrm{E}\left[R(T_{X,n,k})^2\right]$ when H_0 is true, we can determine a precise rejection rule for rejecting H_0 at a chosen significance level. Thus, a consistent estimator of the approximate critical value of τ_{Xn} for a given significance level α can be obtained by the $1-\alpha/2$ quantile of the distribution of $N(0, \mathrm{E}\left[R(T_{X,n,k})^2\right])$. Let c^* be the $1-\alpha/2$ quantile of the sample distribution. One rejects H_0 if:

$$|\tau_{Xn}| \geq c^*.$$

We use τ_{Xn} as the test statistic for testing H_0 and the asymptotic properties of τ_{Xn} are summarized in the following theorem.

Theorem 4.1. *If Assumptions 4.1 and 4.2 hold. Then,*
(i) under H_0,

$$P\left(|\tau_{Xn}| \geq c^*\right) \to 0 \text{ as } n \to \infty.$$

(ii) under a fixed alternative H_1,

$$P\left(|\tau_{Xn}| \geq c^*\right) \to 1 \text{ as } n \to \infty.$$

The critical value c^* for the τ_{Xn} is consistent against a fixed alternative. Therefore, under Assumptions 4.1 and 4.2, rejection of the null H_0 implies the nonparametric family of conditional density functions $\left\{f(x|z) = f_V(x - \mathrm{E}[X|Z=z]) : z \in \mathcal{Z}\right\}$ is complete in $L^1_{bnd}(\mathcal{X})$.

Remark 4.1. *The nonparametric specifications in Assumption 4.1(ii) are related to observable variables* X, Z. *A formal statistical test of the validity of the full independence between* V *and* Z *is possible because we can replace* V *by the residual of an estimator of* $E[X|Z=z]$ *and* X. *Thus, we can use an observable data to justify the maintained assumption.*

Remark 4.2. *In order to fulfill the requirement of an independent error term in the regression form of* X, *we may add more exogenous variables. For example, consider a nonparametric regression model as follows:*

$$y = m(x, z_1) + v, \text{ with } E(v|z) = 0, \quad z = (z_1, z_2) \tag{35}$$

where x *is an endogenous regressor and may be correlated with* v, *and* z_1, z_2 *are exogenous variables. In this case, the relation* $X = E[X|Z=z] + V$, *with* V *is independent of* Z *is more likely to hold than without adding the exogenous variable* z_1. *The completeness in this model is referred to that the nonparametric family of conditional density functions* $\left\{f(x|z) = f_V(x - E[X|Z=z]) : z_2 \in \mathcal{Z}_2\right\}$ *is complete in* $L^1_{bnd}(\mathcal{X})$ *for each* $z_1 \in \mathcal{Z}_1$, *where* $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2$ *is the support of* z.

Remark 4.3. *It is tempting to estimate the conditional mean of* X *given* Z *and verify whether its range fits the whole line for a given dataset. However, one should always be wary of using the sample observations to make a judgement about the support condition could be misleading. For example, the realization of one sample from unbounded normal distributions may appear to indicate that the unbounded support conditions fail.*

Remark 4.4. *Theorem 4.1 shows that for a fixed distribution, the proposed test is* $o(1)$ *under the null and consistent against a fixed alternative and this only establishes pointwise asymptotic size control. This pointwise result does not contradict to the uniform result in Canay, Santos, and Shaikh (2013).*

Remark 4.5. *We have interpreted a rejection of null hypothesis using the proposed tests as evidence of completeness. This is appropriate provided we maintain Assumptions 4.1 and 4.2. However, if Assumption 4.1 is violated then a test for completeness can reject* H_0, *even if the family is not complete. Therefore, it would be better to test for the convolution structure in Assumption 4.1(ii) first, since the specification is important to our result. Then, once we are satisfied with the test for the convolution structure, we can test for completeness.*[5]

[5] We thank the editor Esfandiar Maasoumi to point this out.

4.1.1. Transformation Models

Lemma 2.3 describes the conditions in which the completeness holds under the nonparametric additive models with independent errors in Eq. (5). In this completeness result, one of condition is related to the nonvanishing property of the characteristic function of the independent error V which is unobservable from data. In order to get an estimate of V and apply the proposed test to detect zeros of the characteristic function of V, we consider a transformation model which is encompassed in Eq. (5):

$$X = \Lambda(\beta Z + V), \tag{36}$$

where $\Lambda(\cdot)$ is strictly increasing function. Han (1987) proposes a maximum rank correlation estimator to estimate β consistently. Sherman (1993) establishes the asymptotic normality of the maximum rank correlation estimator. Given a $\sqrt{n}$-consistent estimator for β, Horowitz (1996) and Chen (2002) propose a rank-based estimator for Λ. Because β and Λ are estimable, we also obtain an estimate of V. Denote an estimate of V as $\widehat{V}$.

Assumption 4.3. *(i) suppose Eq. (5) holds; (ii) Range$(m) = \mathbb{R}$; (iii) assume V is independent of Z.*

Under Assumption 4.3, we consider the following hypothesis testing:

$$H_0 : \phi_V(t) = 0 \text{ for some } t. \tag{37}$$

$$H_1 : \phi_V(t) \neq 0 \text{ for all } t. \tag{38}$$

A proposed test statistic is

$$\tau_{\widehat{V}n} = \sqrt{n}a_{\widehat{V},n}(T_{\widehat{V},n,k}), \tag{39}$$

where the notations $a_{\widehat{V},n}$ and $T_{\widehat{V},n,k}$ are the squared modulus of the empirical characteristic function of $\widehat{V}$ and $T_{\widehat{V},n,k}$ is an iterative estimator similar to $T_{X,n,k}$. The rest testing procedure is similar to the procedure for τ_{Xn}. A rejection of the null H_0 implies the nonparametric family of conditional density functions $\left\{f(x|z) : z \in \mathcal{Z}\right\}$ is complete in $L^1_{bnd}(\mathscr{X})$.

4.2. Nonclassical Measurement Error Models with Instrumental Variable

Consider nonclassical measurement error models with the following joint density:

$$f_{YX^*}(y,x^*)$$

where y is the dependent variables, and x^* is the true explanatory variable. However, x^* is not observed, and we observe a measure of x^* by x. Hu and Schennach (2008) rely on the availability of an instrument z to show that the joint distribution f_{YX^*} is identified from knowledge of the distribution of all observed variables Y,X,Z. Hu and Schennach (2008) show the equation

$$f_{YXZ}(y,x,z)=\int f_{YX^*}(y,x^*)f_{X|X^*}(x|x^*)f_{Z|X^*}(z|x^*)dx^*$$

admits a unique solution $(f_{YX^*},f_{X|X^*},f_{Z|X^*})$ for a given observable joint distribution f_{YXZ}. Define the following integral operators:

$$\begin{aligned}
L_{X|X^*}:L^1_{bnd}(\mathscr{X}^*)&\to L^1_{bnd}(\mathscr{X})\\
(L_{X|X^*}h)(x)&=\int h(x^*)f_{X|X^*}(x|x^*)dx^*,\\
L_{Z|X^*}:L^1_{bnd}(\mathscr{X}^*)&\to L^1_{bnd}(\mathscr{Z})\\
(L_{Z|X^*}h)(z)&=\int h(x^*)f_{Z|X^*}(z|x^*)dx^*.
\end{aligned}$$

Hu and Schennach (2008) utilize injectivity of these two integral operators, $L_{X|X^*}$ and $L_{Z|X^*}$ along with other location and normalization conditions to obtain uniqueness of spectral decomposition of an integral operator and provide the identification result. The injectivity of the integral operators, $L_{X|X^*}$ and $L_{Z|X^*}$ implies that the families of conditional density functions $\left\{f_{X|X^*}(x|x^*):x\in\mathscr{X}\right\}$ and $\left\{f_{Z|X^*}(z|x^*):z\in\mathscr{Z}\right\}$ are complete in $L^1_{bnd}(\mathscr{X}^*)$ respectively. The identification result in Hu and Schennach (2008) requires two completeness conditions, the completeness of $f_{X|X^*}$ and the completeness of $f_{Z|X^*}$.

Follow the discussion in Section 2, we have

Proposition 4.1. *Consider $X=m_1(X^*)+E_1$, where m_1 is monotonic and E_1 is independent of X^*. If the characteristic function of X is everywhere nonvanishing, then the nonparametric family of conditional density functions $\left\{f_{X|X^*}(x|x^*)=f_{E_1}(x-m_1(x^*)):x\in\mathscr{X}\right\}$ is complete in $L^1_{bnd}(\mathscr{X}^*)$.*

Assumption 4.4. *Assume $X=m_1(X^*)+E_1$ with m_1 is monotonic and E_1 is independent of X^*.*

Assumption 4.5. *Denote $A_{X0} = \inf\{t > 0 : a_X(t) = 0\}$. (i) Suppose $E(|X|) < \infty$, A_{X0} is an isolated zero of $a_X(t)$, and $a_X(t)$ is smooth in some neighborhood of A_{X0};(ii)*

$$\int\int |x_1 - x_2|^{\alpha} dF_{x_1} dF_{x_2} < \infty. \tag{40}$$

Proposition 4.1 implies a test for an everywhere nonvanishing characteristic function of X under Assumption 4.4 can be regarded as a test for completeness of $f_{X|X^*}$. We propose a test similar to the test in the subsection 4.1. The null hypothesis of the test is

$$H_{X0} : \phi_X(t) = 0 \text{ for some } t. \tag{41}$$

The alternative hypothesis is

$$H_{X1} : \phi_X(t) \neq 0 \text{ for all } t. \tag{42}$$

Proposition 4.1 implies under Assumption 4.4, a rejection of H_{X0} implies $\left\{f_{X|X^*}(x|x^*) = f_{E_1}(x - m_1(x^*)) : x \in \mathscr{X}\right\}$ is complete in $L^1_{bnd}(\mathscr{X}^*)$. In a similar manner as the subsection 4.1, the test statistic is given by:

$$\tau_{Xn} = \sqrt{n} a_{X,n}(T_{X,n,k}), \tag{43}$$

where $a_{X,n}$ is the squared modulus of the empirical characteristic function and $T_{X,n,k}$ is an iterative estimator for finding $A_{X0} = \inf\{t > 0 : a_X(t) = 0\}$ and its definition is in Eq. (22). However, the test statistic only works for an inference for the completeness of $f_{X|X^*}$. To provide an inference for the completeness condition in Hu and Schennach (2008), we also need to incorporate an inference for the completeness of $f_{Z|X^*}$. Thus, our testing strategy consists of a two-step procedure: the first test is an test for the completeness of $f_{X|X^*}$ while the second test is an test for the completeness of $f_{Z|X^*}$.

Assumption 4.6. *Assume $Z = m_2(X^*) + E_2$ with m_2 is monotonic and E_2 is independent of X^*.*

Assumption 4.7. *Denote $A_{Z0} = \inf\{t > 0 : a_Z(t) = 0\}$. (i) Suppose $E(|Z|) < \infty$, A_{Z0} is an isolated zero of $a_Z(t)$, and $a_Z(t)$ is smooth in some neighborhood of A_{Z0};(ii)*

$$\int\int |z_1 - z_2|^{\alpha} dF_{z_1} dF_{z_2} < \infty \tag{44}$$

The null hypothesis of the second test is

$$H_{Z0} : \phi_Z(t) = 0 \text{ for some } t. \tag{45}$$

The alternative hypothesis is

$$H_{Z1} : \phi_Z(t) \neq 0 \text{ for all } t. \tag{46}$$

The second test statistic is given by:

$$\tau_{Zn} = \sqrt{n} a_{Z,n}(T_{Z,n,k}), \tag{47}$$

where $a_{Z,n}$ is the squared modulus of the empirical characteristic function for Z and $T_{Z,n,k}$ is an iterative estimator for finding $A_{Z0} = \inf\{t > 0 : a_Z(t) = 0\}$, where $a_Z(t)$ is the squared modulus of the characteristic function for Z. The definition of $T_{Z,n,k}$ is similar to $T_{X,n,k}$ in Eq. (22).

Under Assumptions 4.4, and 4.5, failure to reject H_{X0} rules out the completeness of $f_{X|X^*}$ asymptotically. Therefore, if we fail to reject the null hypothesis H_{X0}, we stop the testing procedure and decide against the two completeness conditions. On the other hand, if we reject the first null H_{X0}, but fail to reject the second one H_{Z0}, one may still decide against the two completeness conditions. Finally, if the null hypotheses of both tests are rejected, there is evidence for that the two completeness conditions hold. That means that if we reject both H_{X0} and H_{Z0}, under Assumptions 4.4, 4.5, 4.6, and 4.7, both families $\left\{f_{X|X^*}(x|x^*) : x \in \mathscr{X}\right\}$ and $\left\{f_{Z|X^*}(z|x^*) : z \in \mathscr{Z}\right\}$ are complete in $L^1_{bnd}(\mathscr{X}^*)$.

Let c_1^* and c_2^* be the $1-\alpha/2$ quantile of the sample distributions $N(0, \mathrm{E}\left[R(T_{X,n,k})^2\right])$ and $N(0, \mathrm{E}\left[R(T_{Z,n,k})^2\right])$, respectively. Based on the limiting distributions of τ_{Xn} and τ_{Zn}, we device the following two-step decision rule.

Decision Rule:

1. If $|\tau_{Xn}| \leq c_1^*$, we fail to reject H_{X0} and stop the testing procedure. We decide against the two completeness conditions.

2. If $|\tau_{Xn}| \geq c_1^*$ and $|\tau_{Zn}| \leq c_2^*$, we decide against the two completeness conditions.

3. If $|\tau_{Xn}| \geq c_1^*$ and $|\tau_{Zn}| \geq c_2^*$, we decide in favor of the two completeness conditions.

We establish the validity of our testing procedure by the following theorem.

Theorem 4.2. *Suppose Assumptions 4.4, 4.5, 4.6, and 4.7 hold. Then,*
(i) under H_{X0},

$$P\left(|\tau_{Xn}| \geq c_1^*\right) \to 0 \text{ as } n \to \infty.$$

(ii) under $H_{X1} \cap H_{Z0}$ for a fixed alternative H_{X1},

$$P\left(|\tau_{Xn}| \geq c_1^*, |\tau_{Zn}| \geq c_2^*\right) \to 0 \text{ as } n \to \infty.$$

(iii) under $H_{X1} \cap H_{Z1}$ for fixed alternatives H_{X1} and H_{Z1},

$$P\left(|\tau_{Xn}| \geq c_1^*, |\tau_{Zn}| \geq c_2^*\right) \to 1 \text{ as } n \to \infty.$$

Remark 4.6. *The nonparametric specifications in Assumptions 4.4 and 4.6 are related to unobserved variables X^*. In this case, the two assumptions can not be verified empirically as Assumption 4.1 in the nonparametric IV regression model in Section 4.1. To justify the nonparametric specifications, we can present discussions of these two assumptions for a particular empirical application.*

4.3. Nonparametric IV Regression Models Revisited

We reconsider the nonparametric IV regression model from Section 4.1, showing how we can apply the results from previous sections to provide a test on the completeness of the family of conditional density functions $\left\{f(x|z) : z \in \mathcal{Z}\right\}$ without a direct convolution structure between x and z.

Assumption 4.8. *Suppose there exist an exogenous latent variable W^* satisfying (i) $X = m_1(W^*) + V_1$ with $Range(m_1) = \mathbb{R}$ and V_1 is independent of W^*, (ii) $Z = m_2(W^*) + V_2$ with m_2 is monotonic and V_2 is independent of W^*, and (iii) V_1 and V_2 are conditional independent given W^*, i.e., $V_1 \perp\!\!\!\perp V_2 | W^*$.*

Assumption 4.8 is compatible with the validity of the chosen instrumental variable Z. This setup attempts to control the endogeneity by exploiting the common exogenous latent variable W^* in X and Z, and the instrumental variable Z is not be related to the unobserved error U. It follows that $f_{X|Z,W^*} = f_{X|W^*}$.

Assumption 4.9. *(Restrictions on densities) The joint density of x, z, and w^* admits a bounded density and all related marginal and conditional densities are also bounded.*

Proposition 4.2. *Suppose Assumptions 4.8 and 4.9 hold. If the characteristic functions of X and Z are both everywhere nonvanishing, then the nonparametric family of conditional density functions $\left\{f(x|z) : z \in \mathcal{Z}\right\}$ is complete in $L^1_{bnd}(\mathscr{X})$.*

Therefore, we can apply the two-step test procedure in subsection 4.2 using τ_{Xn} in Eq. (43) and τ_{Zn} in Eq. (47) to test the completeness of $\left\{f(x|z) : z \in \mathcal{Z}\right\}$ under Assumptions 4.8 and 4.9. Let c_1^* and c_2^* be the $1-\alpha/2$ quantile of the sample distributions $N(0, \mathrm{E}\left[R(T_{X,n,k})^2\right])$ and $N(0, \mathrm{E}\left[R(T_{Z,n,k})^2\right])$, respectively. Similarly to the two-step test procedure in subsection 4.2, we have the following decision rule.

Decision Rule:

1. If $|\tau_{Xn}| \le c_1^*$, we stop the testing procedure and decide against the completeness condition.

2. If $|\tau_{Xn}| \ge c_1^*$ and $|\tau_{Zn}| \le c_2^*$, we decide against the completeness condition.

3. If $|\tau_{Xn}| \ge c_1^*$ and $|\tau_{Zn}| \ge c_2^*$, we decide in favor of the completeness condition.

Theorem 4.3. *Suppose Assumptions 4.4, 4.5, 4.6, 4.7, 4.8 and 4.9 hold. Then,*
(i) under H_{X0},

$$P\left(|\tau_{Xn}| \ge c_1^*\right) \to 0 \text{ as } n \to \infty.$$

(ii) under $H_{X1} \cap H_{Z0}$ for a fixed alternative H_{X1},

$$P\left(|\tau_{Xn}| \ge c_1^*, |\tau_{Zn}| \ge c_2^*\right) \to 0 \text{ as } n \to \infty.$$

(iii) under $H_{X1} \cap H_{Z1}$ for fixed alternatives H_{X1} and H_{Z1},

$$P\left(|\tau_{Xn}| \ge c_1^*, |\tau_{Zn}| \ge c_2^*\right) \to 1 \text{ as } n \to \infty.$$

4.4. Implementation of the Test Statistic

In this subsection, we provide a detailed description of an algorithm that summarizes the steps in the computation of the proposed test estimator τ_{Xn} in Eq. (39). The algorithm can also be used to compute τ_{Zn} in Eq. (47).

Testing Algorithm for τ_{Xn}

1. Given data $\{x_i : i = 1,...,n\}$ of sample size n, choose α from $(0,1]$ and then compute m_α in Eq. (20).

2. Set $T_{X,n,0} = 0$ and construct the squared modulus of the empirical characteristic function $a_{X,n}(t) = |\phi_{X,n}(t)|^2$. Apply $a_{X,n}(t)$ and m_α in Step 1 to the iterative formula in Eq. (22),

$$T_{X,n,k+1} = T_{X,n,k} + \left(\frac{a_{X,n}(T_{X,n,k})}{2^{1-\alpha} m_\alpha}\right)^{\frac{1}{\alpha}}.$$

3. Calculate the test statistic estimator

$$\tau_{Xn} = \sqrt{n}\, a_{X,n}(T_{X,n,k}),$$

and its variance estimator

(48)
$$V_{Xn}^2 = 2\mathrm{Re}\{\phi_{X,n}(-T_{X,n,k})^2 \phi_{X,n}(2T_{X,n,k}) + \phi_{X,n}(-T_{X,n,k})\phi_{X,n}(T_{X,n,k})\} - 4a_{X,n}(T_{X,n,k})^2.$$

4. Given a significance level α, use $c_\alpha = \Phi^{-1}(1 - \frac{\alpha}{2})$ to construct the critical value $c^* = c_\alpha V_{Xn}$.

5. Compare the test statistic τ_{Xn} to the critical value c^*. If $|\tau_{Xn}| \geq c^*$, reject the null hypothesis $H_0 : \phi_X(t) = 0$ for some t in favor of the alternative hypothesis $H_1 : \phi_X(t) \neq 0$ for all t. Under Assumptions 4.1 and 4.2, rejecting H_0 implies the completeness of $\left\{f(x|z) = f_V(x - \mathrm{E}[X|Z = z]) : z \in \mathcal{Z}\right\}$ in $L^1_{bnd}(\mathscr{X})$. If the test statistic is less than the critical value, do not reject the null hypothesis. Thus, no conclusion has been reached.

5. Monte Carlo Simulation

In this section, we carry out simulation experiments to study the finite sample performance of the proposed test statistic using the Testing Algorithm in subsection 4.4. First, we consider the generation processes for the variable X. Ten specifications of X are considered:

DGP I: $X \sim U(0,1)$,

DGP II: $X \sim U(-1,1)$,

DGP III: $X \sim N(0,1)$,

DGP IV: $X \sim N(1,1)$,

DGP V: $X \sim Gamma(2,2)$,

DGP VI: $X \sim Tri(0,1,0)$

DGP VII: $X \sim Tri(0,1,0.25)$,

DGP VIII: $X \sim Tri(0,1,0.5)$,

DGP IX: $X \sim Trun(N(0,1),[-1,1])$,

DGP X: $X \sim Trun(N(1,1),[-1,1])$,

where the shorthand $Tri(a,b,c)$ is used to indicate that the random variable X has the triangular distribution with the lower limit a, the upper limit b and the mode c and $Trun(N(a,b),[-1,1])$ represents a truncated normal distribution over interval $[-1,1]$ generated by $F_Q^{-1}(u\cdot(F_Q(b)-F_Q(a))+F_Q(a))$, where F_Q is the CDF of the normal distribution $N(a,b)$, F_Q^{-1} is the inverse of F_Q, and u is a uniform random variable on $[0,1]$. We consider sample sizes 500, and 1,000 and for each case, we consider 1000 simulation replications. The estimation results for the proposed test statistics are summarized in Table 1. For DGPs I and II, the rejection rates are very small, which are close to zero. This implies that there is strong evidence that the null hypothesis is likely to hold. That is the characteristic function of X is likely to vanish at some point. This implies that under our nonparametric specifications in Section 4, the families $\left\{f(x|z) = f_V(x - \mathrm{E}[X|Z=z]) : z \in \mathcal{Z}\right\}$ and $\left\{f_{X|X^*}(x|x^*) = f_{E_1}(x - m_1(x^*)) : x \in \mathcal{X}\right\}$ are not likely to be complete. The characteristic functions of the population distributions in DGPs I and II have infinitely many zeros so the proposed test performs well in the DGPs to detect their zeros. For DGPs III, IV and V, the rejection rates are much higher than the nominal size 5% and increase with sample size, indicating that our test are consistent when the null hypothesis is violated. The estimation results are also consistent with the population distributions in DGPs III, IV and V. DGPs VI, VII, VIII, IX, and X are drawn form triangular distributions. While DGP VIII is bounded and symmetric, DGPs VI, VII, IX, and X are bounded and asymmetric. For DGP VIII, the rejection rates are 0.008 and 0.072, which are close to the nominal size 5% given that there exists zeros for its population distribution. For DGPs VI, VII, IX, and X, the rejection rates are much higher than the nominal size 5% indicating a strong evidence against the null hypothesis. DGPs XI, and XII are drawn form truncated normal distributions over interval $[-1,1]$. While DGP XI is symmetric, DGP XII is asymmetric. The rejection rates of the estimation results indicate that the characteristic function of X in DGP XI is likely to vanish at some point and the characteristic function of X in DGP XII does not have any zero.

At a neighborhood of A_0, the squared modulus of the empirical characteristic function $a_{X,n}$ has an associated confidence interval that the true parameter is in the proposed range with some assigned confidence level. Follow the Testing Algorithm in subsection 4.4, the $100(1-\alpha)\%$ confidence interval with $\alpha = 5\%$ at $T_{X,n,k}$ is given by

$$\left(a_{X,n}(T_{X,n,k}) - \frac{c_\alpha V_{Xn}}{\sqrt{n}}, a_{X,n}(T_{X,n,k}) + \frac{c_\alpha V_{Xn}}{\sqrt{n}}\right), \tag{49}$$

The estimation results for the associated confidence intervals in DGPs I, III, VIII, and X are plotted in Figures 1 and 2 for $N = 1000$. These plots show the shapes of the estimates $a_{X,n}$ (blue lines), the location of the first zero estimator $T_{X,n,k}$ (red hexagram), the 2.5th and 97.5th percentile confidence interval at $T_{X,n,k}$ (cyan solid line), the points of zeros (red solid line), and the 2.5th and 97.5th percentile confidence bands of $a_{X,n}$ are in black dashed lines.

In the upper panel of Figures 1 and 2, the confidence interval at $T_{X,n,k}$ contain the zero. This indicates that the possible values of the squared modulus of the empirical characteristic function at 95% significant level contain zero and the proposed test statistics does not reject the null hypothesis, i.e., the characteristic function vanishes at some point. In the bottom panel of Figures 1 and 2, the confidence interval at $T_{X,n,k}$ does not contain the zero. This indicates that it is highly likely that $a_{X,n}$ is bigger than zero and the proposed test statistics rejects the null hypothesis, i.e., the characteristic function does not vanish.

6. Empirical Illustration

This section applies the developed test statistics to a measurement model to illustrate our method empirically. We use a data set that matches self-reported earnings from the Current Population Survey (CPS) to employer-reported social security records (SSR) earnings from 1978 (the CPS/SSR Exact Match File). This data-set has been used by Bound and Krueger (1991), Bollinger (1998), and Chen, Hong, and Tamer (2005). While Bound and Krueger (1991) and Bollinger (1998) use this dataset to argue that the classical measurement error model is not appropriate for reporting errors in male earnings, Chen, Hong, and Tamer (2005) study the problem of parameter inference in econometric models allowing the data are measured with non-classical measurement error. As in Chen, Hong, and Tamer (2005) which assume that the SSR earnings are more accurate, we treat the SSR reporting earnings as the correct earnings data. We will show how one can use the proposed test statistics to make inference on the completeness of the family of conditional density functions $\left\{f_{X|X^*}(x|x^*) : x \in \mathscr{X}\right\}$, where we model the CPS earnings and the SSR earnings as X and X^* respectively.

The population considered here is composed of both men and women, regardless of their age. Following the selection criteria in Chen, Hong, and Tarozzi (2005), we obtain an exact match sample of 38,759 observations. We further divide the data into three education categories: High School or Lower, Some College, and College or Higher. Years of education assigned to each category are 0-12, 13-15, and 16-19, respectively. Table 2 reports summary statistics for the three subsamples. Individuals in High School or Lower are more likely to be older, and not white. They also have smaller CPS and SSR earnings. Individuals in all three groups appear to report higher earnings, i.e., the mean of the CPS earnings are higher than the mean of the SSR earnings. The gap of the means in the CPS and SSR earnings increases with years of education.

Our approach to analyzing the relationship between the CPS earnings and the SSR earnings is to consider the conditional mean function of the CPS earnings on the SSR earnings and the three education categories:

$$\mathrm{E}\left[X|X^*,\ \text{Education}\right]. \tag{50}$$

Denote the residual from this nonparametric regression form as E. Write

$$X = \mathrm{E}\left[X|X^*,\ \text{Education}\right] + E. \tag{51}$$

The conditional mean function represents individual reporting behavior while the residual is a random error. This implies that $f_{X|X^*}(x|x^*,\ \text{Education}) = f_E(x - \mathrm{E}[X|X^* = x^*,\ \text{Education}])$. Bollinger (1998) utilizes Nayadara-Watson kernel-regression estimators to estimate the conditional mean function. We adopt a series method to estimate the conditional mean function using cubic splines with knots at $\mathrm{E}[X^*] - \mathrm{Std}[X^*]$, $\mathrm{E}[X^*]$, and $\mathrm{E}[X^*] + \mathrm{Std}[X^*]$.

Figure 3 presents the estimation results of $\mathrm{E}[X|X^* = x^*,\ \text{Education}]$ at various education categories. In each plot, the blue hexagram line represents the series estimation of the conditional mean function and the red solid line represents the 45° line. If the measurement error were not related to earnings, the blue hexagram line would coincide with the 45° line and the measurement error is classical. In the first plot for the education group, High School or Lower, individuals with more than $8000 SSR earnings are more likely to underreporting their earnings. However, individuals with less than $2000 SSR earnings are in average overreporting their earnings. The overreporting pattern at low SSR earnings also appears in the second plot for the education group, Some College. As for the third plot, all individuals with at least college education are more likely to overreporting their earnings except for individuals with more than $15,000 SSR earnings. In this education category, the overreporting is much severe at low SSR earnings.

Finally, we investigate the zeros of the characteristic functions of the CPS earnings and the residual by applying the proposed test statistic. Table 3 reports the results of the tests on the three education categories. At each education category, the test statistics τ_{Xn} and τ_{En} are bigger than their corresponding critical values with a 5% significant level. This provides strong evidence to reject the null hypothesis that there exists a zero for the characteristic functions of X and E at each education category. If we maintain that $\mathrm{E}[X|X^* = x^*, \text{Education}]$ is monotonic at x^* and the residual E is independent of X^* then the rejection results imply that the family of conditional density functions $\left\{f_{X|X^*}(x|x^*, \text{Education}) = f_E(x - \mathrm{E}[X|X^* = x^*, \text{Education}]) : x \in \mathscr{X}\right\}$ is more likely to be complete in $L^1_{bnd}(\mathscr{X}^*)$ at each education category.

7. Conclusion

This paper has been concerned with hypothesis testing of the completeness condition in a class of nonparametric specification. This study was motivated by a condition for the completeness in Mattner (1993): the nonparametric location family of functions $\{f_V(x-z) : z \in \mathbb{R}\}$ is complete if and only if the characteristic function of V is everywhere nonvanishing. Based on the condition, we present simple test statistics for the completeness in nonparametric IV regression models with or without a convolution structure between the endogenous variable and the instrumental variable, and nonclassical measurement error models with instrumental variables. The test statistics are relatively simple because they are derived from marginal distributions of observables instead of joint distributions. The advantage of the property is that the test statistics can be used to test completeness conditions related to unobservables. We describe asymptotic behavior of the tests under the null and alternative hypotheses and investigate the finite sample properties of the test through a Monte Carlo study. In our empirical illustration, we test for the completeness for a measurement error model of self-reported earnings using data from the CPS/SSR 1978 exact match file. We find evidence for the completeness of the family of conditional density functions $\left\{f_{X|X^*}(x|x^*,\ \text{Education}) = f_E(x - \mathrm{E}[X|X^* = x^*,\ \text{Education}]) : x \in \mathscr{X}\right\}$, where X represents the CPS self-reported earnings and X^* denotes SSR employer-reported earnings.

Our results provide a test for a class of complete distributions other than parametric distributions. On the other hand, Canay, Santos, and Shaikh (2013) consider the hypothesis testing problems for testing completeness in the nonparametric regression model against very general alternatives and show the null hypothesis cannot be tested. Our test establishes only pointwise asymptotic size control, while Canay, Santos, and Shaikh (2013) shows that any test that controls asymptotic size uniformly over a large class of non-complete distributions has trivial asymptotic power against any alternative. Their results imply that our Theorem 4.1 can not be extended to a uniform result, but do not rule out the possibility that a useful test is feasible for a particular class of models. That is how our results are complementary to the results in Canay, Santos, and Shaikh (2013).

One of potential important applications for our results is to provide tests for many nonparametric models involved with deconvolution methods because zero-freeness of the characteristic function is a usual assumption among these approaches. This provides the possibility of data-driven evidence for deconvolution problems.

A. Proofs

Proof of Theorem 3.1: Because $a_X(t) \geq 0$ and $a_{X,n}(t) \geq 0$, by the definition of A_0 and A_n, the squared modulus functions $a_X(t)$ and $a_{X,n}(t)$ attain their minimums at points A_0 and A_n respectively. Since $A_0 < \infty$ is an isolated zero of $a_X(t)$, and the squared modulus functions $a_X(t)$ and $a_{X,n}(t)$ are smooth in some neighborhoods of A_0 and A_n respectively, we can rewrite the definitions of A_0 and A_n as follow:

$$A_0 = \inf\left\{t > 0 : \frac{\partial a_X(t)}{\partial t} = 0, \frac{\partial^2 a_X(t)}{\partial^2 t} > 0\right\}, \tag{52}$$

$$A_n = \inf\left\{t > 0 : \frac{\partial a_{X,n}(t)}{\partial t} = 0, \frac{\partial^2 a_{X,n}(t)}{\partial^2 t} > 0\right\}. \tag{53}$$

For a given $\delta > 0$. By the uniform convergence of $a_{X,n}(t)$ to $a_X(t)$ on each bounded interval in Proposition 3.1 and locally smoothness of $a_{X,n}(t)$ and $a_X(t)$, we have the uniform convergence of $\frac{\partial a_{X,n}(t)}{\partial t}$ to $\frac{\partial a_X(t)}{\partial t}$ on each bounded interval. This implies that $\frac{\partial a_{X,n}(t)}{\partial t} < 0$ almost surely for all $0 < t < A_0 - \delta$ for sufficiently large n. Because $\frac{\partial a_X(t)}{\partial t}$ takes positive values for some points of the interval $(A_0, A_0 + \delta)$, for sufficiently large n, we have $\frac{\partial a_{X,n}(t)}{\partial t} > 0$ almost surely for some point inside $(A_0, A_0+\delta)$. Therefore, for sufficiently large n, $\frac{\partial a_{X,n}(t)}{\partial t} = 0$ for some point in $(A_0-\delta, A_0+\delta)$ or $A_n \in (A_0-\delta, A_0+\delta)$ almost surely. Since δ is arbitrary, this means that $A_n \xrightarrow{a.s.} A_0$ as $n \to \infty$.

Q.E.D.

Proof of Theorem 3.2: Using Proposition 3.2, the distribution limit of $\sqrt{n}\left(\frac{\partial a_{X,n}(t)}{\partial t} - \frac{\partial a_X(t)}{\partial t}\right)$ exists and is normally distributed. Let $U(t)$ be the distribution limit with the variance function $\sigma^2(t)$. By the definition of A_n, we have

$$P\{A_n > r_n\} = P\left\{\frac{\partial a_{X,n}(t)}{\partial t} < 0, \text{ for all } t \le r_n\right\} \tag{54}$$

$$= P\left\{\sqrt{n}\left(\frac{\partial a_{X,n}(t)}{\partial t} - \frac{\partial a_X(t)}{\partial t}\right) < -\sqrt{n}\frac{\partial a_X(t)}{\partial t}, \text{ for all } t \le r_n\right\} \tag{55}$$

$$= P\left\{U(t) < -\sqrt{n}\frac{\partial a_X(t)}{\partial t}, \text{ for all } t \le r_n\right\}. \tag{56}$$

This implies

$$P\{A_n > r_n\} \le P\left\{U(r_n) < -\sqrt{n}\frac{\partial a_X(r_n)}{\partial t}\right\} = \Phi\left(-\sqrt{n}\frac{\frac{\partial a_X(r_n)}{\partial t}}{\sigma(r_n)}\right). \tag{57}$$

By the choice of r_n, as $n \to \infty$ we have

$$\sqrt{n}\frac{\frac{\partial a_X(r_n)}{\partial t}}{\sigma(r_n)} = \sqrt{n}\frac{\frac{\partial a_X(r_n)}{\partial t} - \frac{\partial a_X(A_0)}{\partial t}}{\sigma(r_n)} = \frac{\sqrt{n}a''_X(A_0)(r_n - A_0)}{\sigma(A_0)} + o(1) = z + o(1). \tag{58}$$

Plugging the relation back to Eq. (57) yields

$$P\{A_n \le r_n\} \ge \Phi(z) \text{ as } n \to \infty. \tag{59}$$

On the other hand, for the reverse inequality, we consider

$$P\{A_n > r_n\} \tag{60}$$

$$= P\left\{U(t) < -\sqrt{n}\frac{\partial a_X(t)}{\partial t}, \forall t \le r_n\right\} \tag{61}$$

$$= P\left\{U(t) < -\sqrt{n}\frac{\partial a_X(t)}{\partial t}, \forall t \le \nu_n\right\} \bigcap P\left\{U(t) < -\sqrt{n}\frac{\partial a_X(t)}{\partial t}, \forall \nu_n \le t \le r_n\right\}, \tag{62}$$

where $\nu_n = A_0 - \frac{\varepsilon_n}{\sqrt{n}}$ with $\varepsilon_n \to \infty$ but $\varepsilon_n = o(\sqrt{n})$. For the first term in Eq. (62), consider its complement

$$P\left\{U(s) \geq -\sqrt{n}\frac{\partial a_X(s)}{\partial t}, \exists s \leq v_n\right\}, \tag{63}$$

Because $\frac{\frac{\partial a_X(t)}{\partial t}}{\sigma(t)}$ is increasing in t, for sufficiently large n, $\sqrt{n}\frac{\frac{\partial a_X(t)}{\partial t}}{\sigma(t)} < \sqrt{n}\frac{\frac{\partial a_X(v_n)}{\partial t}}{\sigma(v_n)} = \sqrt{n}\frac{a''_X(A_0)(v_n - A_0)}{\sigma(v_n)} = -\frac{a''_X(A_0)\varepsilon_n}{\sigma(v_n)}$ and $\frac{a''_X(A_0)\varepsilon_n}{\sigma(v_n)}$ approaches to ∞ as $n \to \infty$. It follows that

$$P\left\{\frac{U(s)}{\sigma(s)} \geq -\sqrt{n}\frac{\frac{\partial a_X(s)}{\partial t}}{\sigma(s)}, \exists s \leq v_n\right\} \leq P\left\{\frac{U(s)}{\sigma(s)} \geq \frac{a''_X(A_0)\varepsilon_n}{\sigma(v_n)}, \exists s \leq v_n\right\}, \tag{64}$$

By Fernique's Lemma in Marcus (1970) and $\varepsilon_n \to \infty$, this last probability is bounded by $c_2 \exp(-c_1\varepsilon_n^2) \to 0$ as $n \to \infty$ with some positive constants c_1, and c_2. Therefore, as $n \to \infty$, we obtain

$$P\left\{U(t) < -\sqrt{n}\frac{\partial a_X(t)}{\partial t}, \forall t \leq v_n\right\} \to 1. \tag{65}$$

As for the second term in Eq. (62), for any $\delta > 0$, $\sup_{v_n \leq t \leq r_n}\left|\frac{U(t)}{\sigma(t)} - \frac{U(r_n)}{\sigma(r_n)}\right| \leq \delta$ for sufficiently large n, and then we have $\frac{U(t)}{\sigma(t)} \leq \frac{U(r_n)}{\sigma(r_n)} + \delta$ for sufficiently large n. This implies

$$\begin{aligned}
&P\left\{U(t) < -\sqrt{n}\frac{\partial a_X(t)}{\partial t}, \forall v_n \leq t \leq r_n\right\} && (66)\\
&= P\left\{\frac{U(t)}{\sigma(t)} < -\sqrt{n}\frac{\frac{\partial a_X(t)}{\partial t}}{\sigma(t)}, \forall v_n \leq t \leq r_n\right\} && (67)\\
&\geq P\left\{\frac{U(r_n)}{\sigma(r_n)} + \delta < -z + o(1), \sup_{v_n \leq t \leq r_n}\left|\frac{U(t)}{\sigma(t)} - \frac{U(r_n)}{\sigma(r_n)}\right| \leq \delta\right\} && (68)\\
&\geq \Phi(-z + o(1) - \delta) - P\left\{\sup_{v_n \leq t \leq r_n}\left|\frac{U(t)}{\sigma(t)} - \frac{U(r_n)}{\sigma(r_n)}\right| > \delta\right\}. && (69)
\end{aligned}$$

By Theorem 2.1 of Berman (1974), for $0 < \alpha' < 1$ we have

$$P\left\{\sup_{v_n \leq t \leq r_n}\left|\frac{U(t)}{\sigma(t)} - \frac{U(r_n)}{\sigma(r_n)}\right| > \delta\right\} \leq const. \times \left(1 - \Phi(const. \times \delta \times (r_n - v_n)^{-\alpha'/2})\right) \to 0 \tag{70}$$

as $n \to \infty$. Combining the results for any $\delta > 0$, we have proved the reverse inequality and then the statement in Eq. (17) follows. *Q.E.D.*

Proof of Theorem 3.3: (i) Let b_n be any number in $(0, A_n)$. Because $b_n < A_n$, by the definition of A_n we have $a_{X,n}(t) > 0$ for $0 \le t \le b_n$. Then set

$$\Delta_n = \inf\left\{\left(\frac{a_{X,n}(t)}{2^{1-\alpha}m_\alpha}\right)^{\frac{1}{\alpha}} : 0 \le t \le b_n\right\}. \tag{71}$$

It follows that $\Delta_n > 0$ and $T_{X,n,k}$ is a monotone increasing sequence which is bounded by A_n. Since $T_{X,n,k} > b_n$ for $k > [\frac{b_n}{\Delta_n}] + 1$, where $[x]$ denotes the integer part of x, we obtain $T_{X,n,k} \xrightarrow{a.s.} A_n$ as $k \to \infty$.

(ii) Let $\delta > 0$ be given. By the definition of A_0 in Eq. (52) and the continuity of $a_X(t)$, we can choose $\varepsilon > 0$ such that

$$\inf\{t \ge 0 : a_X(t) = \varepsilon\} > A_0 - \delta/2, \tag{72}$$

$$\inf\{t \ge A_0 : a_X(t) = \varepsilon\} < A_0 + \delta/2. \tag{73}$$

Let B be a compact set of the real line containing $[0, A_0 + \delta/2]$. By the uniform convergence of $a_{X,n}(t)$ to $a_X(t)$ on each bounded interval in Proposition 3.1, there exists an N_1 such that for all $n > N_1$,

$$0 \le a_{X,n}(t) \le a_X(t) + \varepsilon \tag{74}$$

uniformly in $t \in B$ with probability as close to one. Therefore, with probability as close to one, for all $n > N_1$,

$$A_0 - \delta/2 \le A_n \le A_0 + \delta/2. \tag{75}$$

The moment condition in Eq. (23) implies that there exists an N_2 such that, with probability as close to one, for all $n > N_2$

$$m_\alpha \le \int\int |x_1 - x_2|^\alpha dF_1 dF_2 + \varepsilon \equiv \mu_\alpha + \varepsilon, \tag{76}$$

by the strong law of large numbers. Combining these two results, with probability as close to one, for $n > N = \max\{N_1, N_1\}$, we obtain

$$\varepsilon_N = \inf\left\{\left(\frac{a_X(t) - \varepsilon}{2^{1-\alpha}(\mu_\alpha + \varepsilon)}\right)^{\frac{1}{\alpha}} : 0 \le t \le A_0 - \delta/2\right\} \tag{77}$$

$$\leq \inf\left\{\left(\frac{a_{X,n}(t)}{2^{1-\alpha}m_\alpha}\right)^{\frac{1}{\alpha}} : 0 \leq t \leq A_n - \delta\right\} \tag{78}$$

because $\{0 \leq t \leq A_n - \delta\} \subset \{0 \leq t \leq A_0 - \delta/2\}$. By the choice of ε, we have $\varepsilon_N > 0$. This implies that with probability as close to one, for all $k > [\frac{A_0-\delta/2}{\varepsilon_N}] + 1$,

$$|T_{X,n,k} - A_n| \leq \delta. \tag{79}$$

(iii) This part follows from the derivation in (ii) by choosing $\delta = \frac{1}{\sqrt{n}}$ and k correspondingly.

Q.E.D.

Proof of Theorem 4.1: Recall that under H_0 and Assumption 4.2, the results in Theorem 3.3(iii) and Theorem 3.2 hold. Consider

$$\tau_{Xn} = \sqrt{n}\left(a_{X,n}(T_{X,n,k}) - a_{X,n}(A_n)\right) + \sqrt{n}\left(a_{X,n}(A_n) - a_{X,n}(A_0)\right) + \sqrt{n}\left(a_{X,n}(A_0) - a_X(A_0)\right) \tag{80}$$

$$= \frac{\partial a_X(T^*_{X,n,k})}{\partial t}\sqrt{n}\left(T_{X,n,k} - A_n\right) + \frac{\partial a_X(A^*_n)}{\partial t}\sqrt{n}(A_n - A_0) + \sqrt{n}\left(a_{X,n}(A_0) - a_X(A_0)\right), \tag{81}$$

where $T^*_{X,n,k}$ is a value between $T_{X,n,k}$ and A_n and A^*_n is a value between A_n and A_0. As $n \to \infty$, we have $\frac{\partial a_X(T^*_{X,n,k})}{\partial t} \to \frac{\partial a_X(A_0)}{\partial t} = 0$ and $\frac{\partial a_X(A^*_n)}{\partial t} \to \frac{\partial a_X(A_0)}{\partial t} = 0$. Because Theorem 3.3(iii) and Theorem 3.2 imply $\sqrt{n}\left(T_{X,n,k} - A_n\right)$ and $\sqrt{n}(A_n - A_0)$ are bounded in probability, as $n \to \infty$, we obtain

$$\tau_{Xn} = \sqrt{n}\left(a_{X,n}(A_0) - a_X(A_0)\right) + o_p(1). \tag{82}$$

By using Proposition 3.2, we have proved $\tau_{Xn} \xrightarrow{d} N(0, \mathrm{E}\left[R(A_0)^2\right])$ with $R(A_0) = 0$[6]. This gives the desired statement at (i). As for the statement at (ii), consider

$$\tau_{Xn} = \sqrt{n}\left(a_{X,n}(T_{X,n,k}) - a_X(T_{X,n,k})\right) + \sqrt{n}a_X(T_{X,n,k}). \tag{83}$$

By Proposition 3.2, the first term is $\xrightarrow{d} N(0, \mathrm{E}\left[R(T_{X,n,k})^2\right])$ which is bounded in probability. But the second term diverges to ∞ under H_1. This proves the statement at (ii).

Q.E.D.

Proof of Proposition 4.2: Because the characteristic function of X is everywhere nonvanishing, under Assumption 4.8(i), Propositions 2.1 implies the nonparametric family of conditional density functions $\left\{f(x|w^*) = f_{V_1}(x - m_1(w^*)) : w^* \in \mathcal{W}^*\right\}$ is complete in $L^1_{bnd}(\mathcal{X})$. On the

[6] Eq. 11 and the fact that $\phi_X(A_0) = \phi_X(-A_0) = a_X(A_0)$ implies that $R(A_0) = 0$.

other hand, by Propositions 4.1 if the characteristic function of Z is everywhere nonvanishing, then under Assumption 4.8(ii), the nonparametric family of conditional density functions $\left\{f(z|w^*) = f_{V_2}(z - m_2(w^*)) : z \in \mathcal{Z}\right\}$ is complete in $L^1_{bnd}(\mathcal{W}^*)$.

With the relation $f_{X|Z,W^*} = f_{X|W^*}$ from Assumption 4.8, by the law of the total probability, we write

$$
\begin{aligned}
f_{XZ}(x,z) &= \int_{\mathcal{W}^*} f_{X|ZW^*}(x|z,w^*) f_{Z|W^*}(z|w^*) f_{W^*}(w^*) dw^* \\
&= \int_{\mathcal{W}^*} f_{X|W^*}(x|w^*) f_{Z|W^*}(z|w^*) f_{W^*}(w^*) dw^*.
\end{aligned} \tag{84}
$$

Suppose that there exists $h \in L^1_{bnd}(\mathcal{X})$ such that $\int_{\mathcal{X}} f_{X|Z}(x|z)h(x)dx = 0$ for a.e. $z \in \mathcal{Z}$. Multiplying both sides of the equation by $f(z)$ yields

$$
\int_{\mathcal{X}} f_{XZ}(x,z)h(x)dx = 0 \text{ for a.e. } z \in \mathcal{Z}. \tag{85}
$$

Plugging the expression of f_{XZ} in Eq. (84) into Eq. (85) yields

$$
\int_{\mathcal{X}} \left(\int_{\mathcal{W}^*} f_{X|W^*}(x|w^*) f_{Z|W^*}(z|w^*) f_{W^*}(w^*) dw^* \right) h(x) dx = 0 \text{ for a.e. } z \in \mathcal{Z}. \tag{86}
$$

Interchanging the integrations, we obtain

$$
\int_{\mathcal{W}^*} \left(\int_{\mathcal{X}} f_{X|W^*}(x|w^*) h(x) dx \right) f_{W^*}(w^*) f_{Z|W^*}(z|w^*) dw^* = 0 \text{ for a.e. } z \in \mathcal{Z}. \tag{87}
$$

By Assumption 4.9, $\left(\int_{\mathcal{X}} f_{X|W^*}(x|w^*)h(x)dx\right) f_{W^*}(w^*)$ is a function in $L^1_{bnd}(\mathcal{W}^*)$. Since the family of conditional density functions $\left\{f(z|w^*) = f_{V_2}(z - m_2(w^*)) : z \in \mathcal{Z}\right\}$ is complete in $L^1_{bnd}(\mathcal{W}^*)$, we have $\left(\int_{\mathcal{X}} f_{X|W^*}(x|w^*)h(x)dx\right) f_{W^*}(w^*) = 0$ for a.e. $w^* \in \mathcal{W}^*$. This implies that $\int_{\mathcal{X}} f_{X|W^*}(x|w^*)h(x)dx = 0$ for a.e. $w^* \in \mathcal{W}^*$. Because $h \in L^1_{bnd}(\mathcal{X})$ and the nonparametric family of conditional density functions $\left\{f(x|w^*) = f_{V_1}(x - m_1(w^*)) : w^* \in \mathcal{W}^*\right\}$ is complete in $L^1_{bnd}(\mathcal{X})$, we obtain $h(x) = 0$ for a.e. $x \in \mathcal{X}$. We have reached the desired result that the nonparametric family of conditional density functions $\left\{f(x|z) : z \in \mathcal{Z}\right\}$ is complete in $L^1_{bnd}(\mathcal{X})$. *Q.E.D.*

References

AI, C., AND X. CHEN (2003): "Efficient Estimation of Models with Conditional Moment Restrictions Containing Unknown Functions," *Econometrica*, 71(6), 1795–1843.

ANDREWS, D. (2017): “Examples of L^2-complete and Boundedly-complete Distributions,” *Journal of Econometrics*, 199(2), 213–220.

BERMAN, S. M. (1974): “Sojourns and Extremes of Gaussian Processes,” *The Annals of Probability*, pp. 999–1026.

BLUNDELL, R., X. CHEN, AND D. KRISTENSEN (2007): “Semi-nonparametric IV Estimation of Shape-Invariant Engel Curves,” *Econometrica*, 75(6), 1613.

BOLLINGER, C. (1998): “Measurement Error in the Current Population Survey: A Nonparametric Look,” *Journal of Labor Economics*, 16(3), 576–594.

BOUND, J., AND A. KRUEGER (1991): “The Extent of Measurement Error in Longitudinal Earnings Data: Do Two Wrongs Make a Right?,” *Journal of Labor Economics*, 9(1), 1–24.

CANAY, I., A. SANTOS, AND A. SHAIKH (2013): “On the Testability of Identification in Some Nonparametric Models with Endogeneity,” *Econometrica*, 81(6), 2535–2559.

CARROLL, R., X. CHEN, AND Y. HU (2010): “Identification and Estimation of Nonlinear Models Using Two Samples with Nonclassical Measurement Errors,” *Journal of Nonparametric Satistics*, 22(4), 379–399.

CHEN, S. (2002): “Rank Estimation of Transformation Models,” *Econometrica*, 70(4), 1683–1697.

CHEN, X., H. HONG, AND E. TAMER (2005): “Measurement Error Models with Auxiliary Data,” *The Review of Economic Studies*, 72(2), 343–366.

CHEN, X., H. HONG, AND A. TAROZZI (2005): “Semiparametric Efficiency in GMM Models with Nonclassical Measurement Error,” *Unpublished Manuscript, Duke Univesity*.

CHEN, X., AND Y. HU (2006): “Identification and Inference of Nonlinear Models Using Two Samples With Arbitrary Measurement Errors,” *Cowles Foundation Discussion Paper No. 1590*.

CHERNOZHUKOV, V., AND C. HANSEN (2005): “An IV Model of Quantile Treatment Effects,” *Econometrica*, 73(1), 245–261.

CHERNOZHUKOV, V., G. IMBENS, AND W. NEWEY (2007): “Instrumental Variable Estimation of Nonseparable Models,” *Journal of Econometrics*, 139(1), 4–14.

DAROLLES, S., Y. FAN, J. FLORENS, AND E. RENAULT (2011): "Nonparametric Instrumental Regression," *Econometrica*, 79(5), 1541–1565.

DAROLLES, S., J. FLORENS, AND E. RENAULT (2006): "Nonparametric Instrumental Regression," .

D'HAULTFOEUILLE, X. (2011): "On the Completeness Condition in Nonparametric Instrumental Problems," *Econometric Theory*, 1, 1–12.

FEUERVERGER, A., AND R. A. MUREIKA (1977): "The Empirical Characteristic Function and its Applications," *The Annals of Statistics*, 5(1), 88–97.

FREYBERGER, J. (2017): "On Completeness and Consistency in Nonparametric Instrumental Variable Models," *Econometrica*, 85, 1629–1644.

HAN, A. K. (1987): "Non-Parametric Analysis of a Generalized Regression Model: The Maximum Rank Correlation Estimator," *Journal of Econometrics*, 35(2-3), 303–316.

HEATHCOTE, C., AND J. HÜSLER (1990): "The First Zero of an Empirical Characteristic Function," *Stochastic Processes and their Applications*, 35(2), 347–360.

HOROWITZ, J. (1996): "Semiparametric Estimation of a Regression Model with an Unknown Transformation of the Dependent Variable," *Econometrica*, pp. 103–137.

HOROWITZ, J. (2011): "Applied Nonparametric Instrumental Variables Estimation," *Econometrica*, 79(2), 347–394.

HOROWITZ, J., AND S. LEE (2007): "Nonparametric Instrumental Variables Estimation of a Quantile Regression Model," *Econometrica*, 75(4), 1191–1208.

HOROWITZ, J. L. (2012): "Specification Testing in Nonparametric Instrumental Variables Estimation," *Journal of Econometrics*, 167(2), 383–396.

HU, Y., AND S. SCHENNACH (2008): "Instrumental Variable Treatment of Nonclassical Measurement Error Models," *Econometrica*, 76(1), 195–216.

HU, Y., S. SCHENNACH, AND J. SHIU (2017): "Injectivity of a Class of Integral Operators with Compactly Supported Kernels," *Journal of Econometrics*, 200(1), 48–58.

HU, Y., AND J. SHIU (2018): "Nonparametric Identification Using Instrumental Variables: Sufficient Conditions for Completeness," *Econometric Theory*, 34(3), 659–693.

MARCUS, M. (1970): "A Bound for the Distribution of the Maximum of Continuous Gaussian Processes," *The Annals of Mathematical Statistics*, 41(1), 305–309.

MATTNER, L. (1993): "Some Incomplete but Boundedly Complete Location Families," *The Annals of Statistics*, 21, 2158–2162.

MUROTA, K., AND K. TAKEUCHI (1981): "The Studentized Empirical Characteristic Function and its Application to Test for the Shape of Distribution," *Biometrika*, 68(1), 55–65.

NEWEY, W., AND J. POWELL (2003): "Instrumental Variable Estimation of Nonparametric Models," *Econometrica*, 71(5), 1565–1578.

ROBIN, J.-M., AND R. J. SMITH (2000): "Tests of Rank," *Econometric Theory*, 16(2), 151–175.

SERFLING, R. J. (2009): *Approximation Theorems of Mathematical Statistics*. John Wiley & Sons.

SHERMAN, R. P. (1993): "The limiting Distribution of the Maximum Rank Correlation Estimator," *Econometrica*, 61, 123–137.

SHIU, J., AND Y. HU (2013): "Identification and Estimation of Nonlinear Dynamic Panel Data Models with Unobserved Covariates," *Journal of Econometrics*, 175(2), 116–131.

WELSH, A. (1986): "Implementing Empirical Characteristic Function Procedures," *Statistics & Probability Letters*, 4(2), 65–67.

Table 1: Test for Non-vanishing C.F.: Empirical Size

	N=500	N=1000	Distributions
DGP I:	0	0	$U(0,1)$
DGP II:	0.001	0	$U(-1,1)$
DGP III:	0.304	0.913	$N(0,1)$
DGP IV:	0.315	0.915	$N(1,1)$
DGP V:	1	1	$Gamma(2,2)$
DGP VI:	0.999	1	$Tri(0,1,0)$
DGP VII:	0.943	1	$Tri(0,1,0.25)$
DGP VIII:	0.008	0.072	$Tri(0,1,0.5)$
DGP IX:	0.955	1	$Tri(0,1,0.75)$
DGP X:	1	1	$Tri(0,1,1)$
DGP XI:	0	0	$Trun(N(0,1),[-1,1])$
DGP XII:	1	1	$Trun(N(1,1),[-1,1])$

Note: Empirical size refers to the fraction of rejections when using the critical value corresponding to a 5% significant level. Only the uniform, symmetric triangular, and symmetric truncated normal distributions fail to satisfy the non-vanishing property for their characteristic functions.

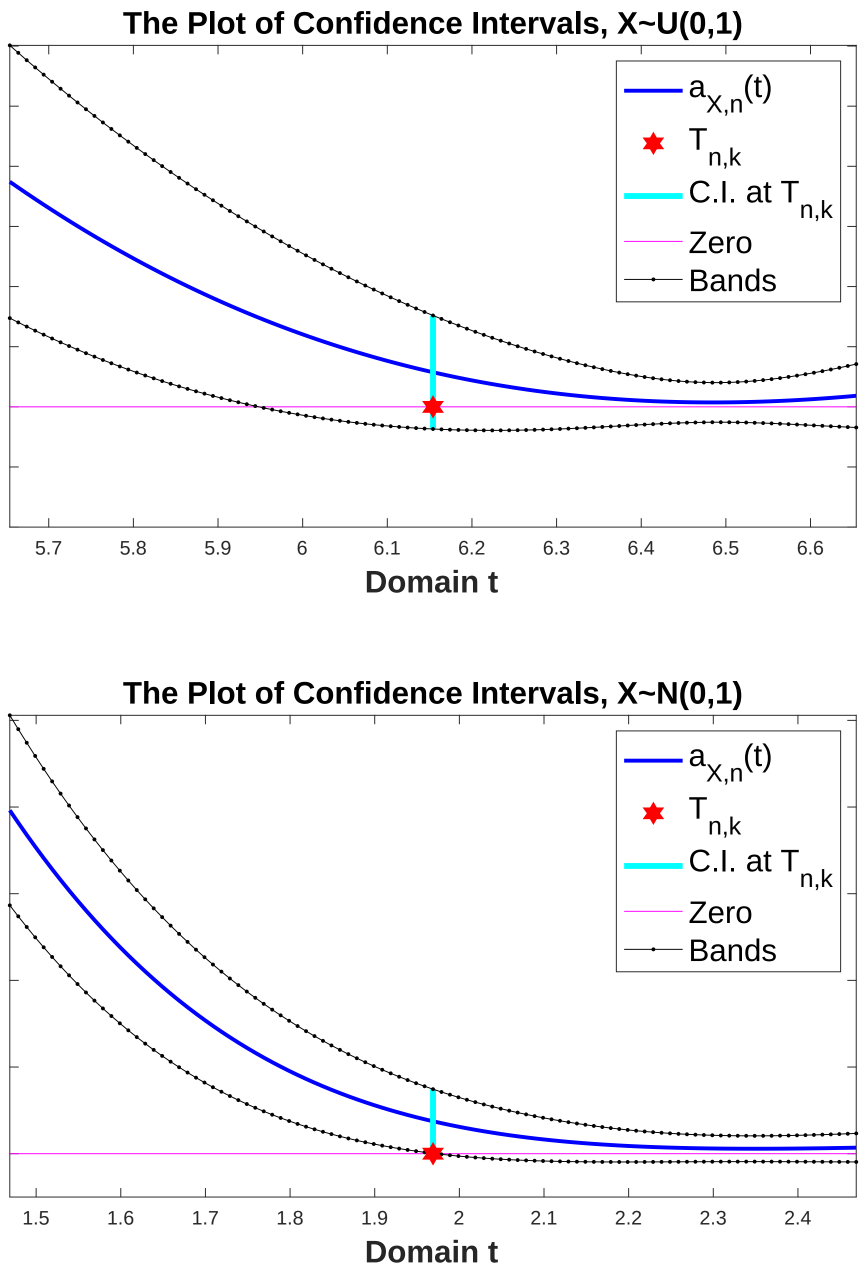
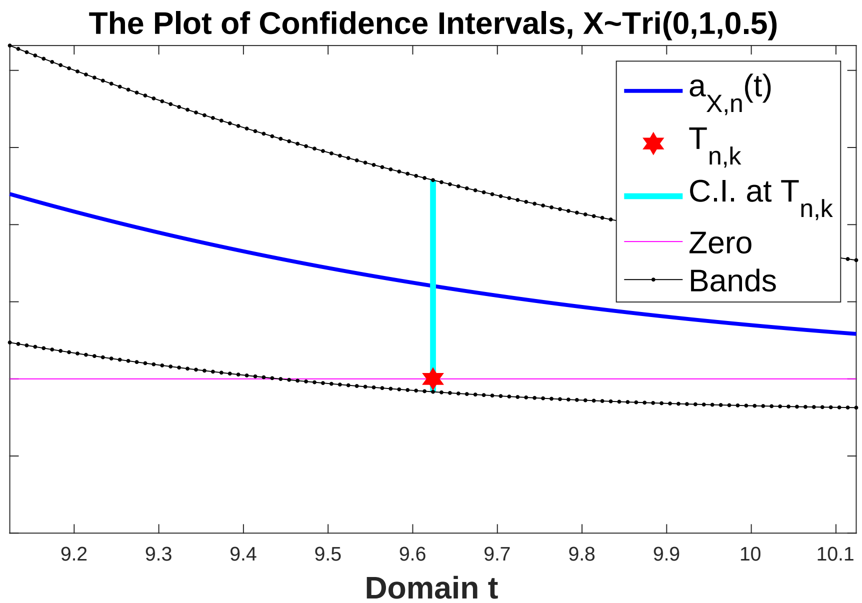


Figure 1: The Illustration of the Confidence Bands of $a_{X,n}$ and the Estimator of A_n, $T_{X,n,k}$



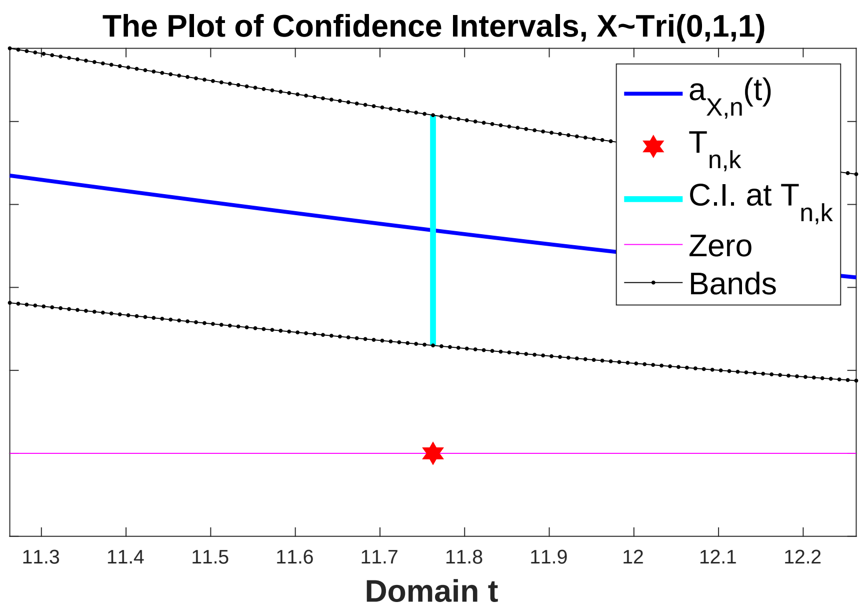


Figure 2: The Illustration of the Confidence Bands of $a_{X,n}$ and the Estimator of A_n, $T_{X,n,k}$

Table 2: Descriptive Statistics of CPS/SSR Exact Match 1978

	High School or Lower		Some College		College or Higher	
	Mean	Std.	Mean	Std.	Mean	Std.
CPS Earnings (<\$16.5)	7.043	6.075	8.980	7.097	13.496	10.535
SSR Earnings (<\$16.5)	6.865	5.322	8.046	5.425	9.994	5.891
Years of Education	9.885	2.193	13.452	0.750	17.378	1.018
Age	41.606	16.310	35.153	13.865	35.487	12.035
Non-white	0.162	0.368	0.091	0.288	0.076	0.265
Married	0.635	0.481	0.629	0.483	0.643	0.479
Sample Size	9,045		21,931		7,783	

Note: All earnings are expressed in a thousand of dollars in 1977. The earnings in the SSR data are capped at the social security maximum of $16,500.

Table 3: Test Statistics for Zeros of Characteristic Functions

	High School or Lower	Some College	College or Higher
τ_{Xn}	2.528	1.850	0.386
c_X^*	0.455	0.311	0.180
τ_{En}	2.110	1.927	1.165
c_E^*	0.421	0.320	0.320

Note: The significance level of the critical values c_X^* and c_E^* are 5%.

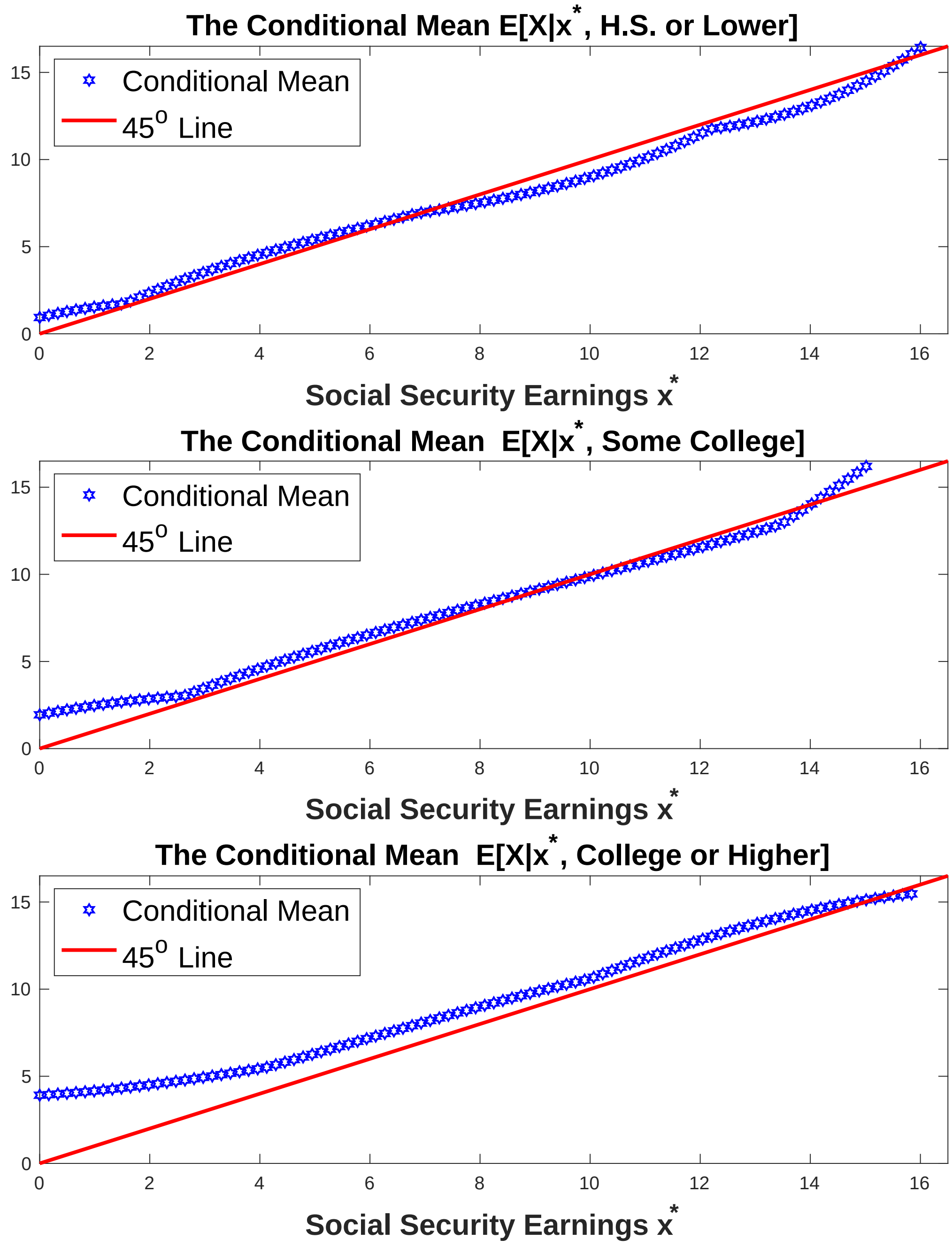


Figure 3: The Series Estimation of the Conditional Mean $\mathrm{E}[X|X^* = x^*,\ \text{Education}]$