Identifying Dynamic Games with Serially-Correlated Unobservables*

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Abstract

In this paper we consider the nonparametric identification of Markov dynamic games models in which each firm has its own unobserved state variable, which is persistent over time. This class of models includes most models in the Ericson and Pakes (1995) and Pakes and McGuire (1994) framework. We provide conditions under which the joint Markov equilibrium process of the firms' observed and unobserved variables can be nonparametrically identified from data. For stationary continuous action games, we show that only three observations of the observed component are required to identify the equilibrium Markov process of the dynamic game. When agents' choice variables are discrete, but the unobserved state variables are continuous, four observations are required.

1 Introduction

In this paper, we consider nonparametric identification in Markovian dynamic games models where which each agent may have its own serially-correlated unobserved state variable. This class of models includes most models in the Ericson and Pakes (1995) and Pakes and McGuire (1994) framework. These models have been the basis for much of the recent

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empirical applications of dynamic game models. Throughout, by "unobservable", we mean variables which are commonly observed by all agents, and condition their actions, but are unobserved by the researcher.

Consider a dynamic duopoly game in which two firms compete. It is straightforward to extend our assumptions and arguments to the case of N firms. A dynamic duopoly is described by the sequence of variables $\left(W_{t+1}, X_{t+1}^*\right), \left(W_t, X_t^*\right), ..., \left(W_1, X_1^*\right)$ where

$$\begin{aligned} W_t &= \left(W_{1,t}, W_{2,t}\right), \\ X_t^* &= \left(X_{1,t}^*, X_{2,t}^*\right). \end{aligned}$$

$W_{i,t}$ stands for the observed information on firm i and $X_{i,t}^*$ denote the unobserved heterogeneity of firm i at period t, which we allow to vary over time and be serially-correlated.

In empirical dynamic games model, the observed variables $W_{i,t}$ consists of two variables:

$$W_{i,t} \equiv (Y_{i,t}, M_{i,t}),$$

where $Y_{i,t}$ denotes firm i's choice, or control variable in period t, and $M_{i,t}$ denotes the state variables of firm i which are observed by both the firms and the researcher. We assume that the serially-correlated variables $X_{1,t}^*$ and $X_{2,t}^*$ are observed by both firms prior to making their choices of $Y_{1,t}, Y_{2,t}$ in period t, but the researcher never observes X_t^*. For simplicity, we assume that each firm's variables $Y_{i,t}, M_{i,t}, X_{i,t}^*$ are scalar-valued.

Main Results: Our goal is to identify the density

$$f_{W_t, X_t^* | W_{t-1}, X_{t-1}^*}, \tag{1}$$

which corresponds to the equilibrium transition density of the choice and state variables along the Markov equilibrium path of the dynamic game. In Markovian dynamic settings, the transition density can be factored into two components of interest:

$$\begin{aligned} f_{W_t, X_t^* | W_{t-1}, X_{t-1}^*} &= f_{Y_t, M_t, X_t^* | Y_{t-1}, M_{t-1}, X_{t-1}^*} \\ &= \underbrace{f_{Y_t | M_t, X_t^*}}_{\text{CCP}} \cdot \underbrace{f_{M_t, X_t^* | Y_{t-1}, M_{t-1}, X_{t-1}^*}}_{\text{state transition}}. \end{aligned} \tag{2}$$

The first term denotes the conditional choice probabilities (CCP) for the firms' actions in period t, conditional on the current state (M_t, X_t^*). In the Markov equilibrium, firms' optimal strategies typically depends just on the current state variables (M_t, X_t^*), but not past values. The second term denotes is the equilibrium Markovian transition probabilities for the state variables (M_t, X_t^*). As shown in Hotz and Miller (1993) and Magnac and Thesmar (2002), once these two structural components are known, it is possible to recover the "deep" structural elements of the model, including the period utility functions.

In an earlier paper (Hu and Shum (2008)), we focused on nonparametric identification of Markovian single-agent dynamic optimization models. There, we showed that in stationary models, four periods of data $W_{t+1}, \ldots, W_{t-2}$ were enough to identify the Markov transition $W_t, X_t^* | W_{t-1}, X_{t-1}^*$, while five observations $W_{t+1}, \ldots, W_{t-3}$ were required for the nonstationary case. In this paper, we focus on Markovian dynamic games. We show that, once additional features of the dynamic optimization framework are taken into account, only three observations $W_t, \ldots, W_{t-2}$ are required to identify $W_t, X_t^* | W_{t-1}, X_{t-1}^*$ in the stationary case, when Y_t is a continuous choice variable. If Y_t is a discrete choice variable (while X_t^* is continuous), then four observations are required for identification.

Related literature Recently, there has been a growing literature related to identification and estimation of dynamic games. Papers include Aguirregabiria and Mira (2007), Pesendorfer and Schmidt-Dengler (2008), Bajari, Benkard, and Levin (2007), Pakes, Ostrovsky, and Berry (2007), and Bajari, Chernozhukov, Hong, and Nekipelov (2007). Our main contribution relative to this literature is to provide nonparametric identification results for the case where there are firm-specific unobserved state variables, which are serially correlated over time. Allowing for firm-specific and serially-correlated unobservables is important, because the dynamic game models in Ericson and Pakes (1995) and Pakes and McGuire (1994) (see also Doraszelski and Pakes (2007)), which provide an important framework for much of the existing empirical work in dynamic games, explicitly contain firm-specific "product quality" variables which are typically unobserved by researchers. The class of models considered in this paper also resemble models analyzed in the dynamic treatment effects literature in labor economics (eg. Cunha, Heckman, and Schennach (2006), Abbring and Heckman (2007), Heckman and Navarro (2007)).

A few recent papers have considered estimation methodologies for games with serially-correlated unobservables. Arcidiacono and Miller (2006) develop an EM-algorithm for estimating dynamic games where the unobservables are assumed to follow a discrete Markov process. Siebert and Zulehner (2008) extend the Bajari, Benkard, and Levin (2007) approach to estimate a dynamic product choice game for the computer memory industry where each firm experiences a serially-correlated productivity shock. Finally, Blevins (2008) develops simulation estimators for dynamic games with serially-correlated unobservables, utilizing state-of-the-art recursive importance sampling ("particle filtering") techniques. However, all these papers focus on estimation of parametric models in which the parameters are assumed to be identified, whereas this paper concerns nonparametric identification.

2 Examples of Dynamic Duopoly Games

To make things concrete, we present two examples of a dynamic duopoly problem, both of which are in the "dynamic investment" framework of Ericson and Pakes (1995) and Pakes and McGuire (1994), but simplified without an entry decision.

Example 1 is a model of learning by doing in a durable goods market, similar to Benkard (2004). There are two heterogeneous firms $i = 1, 2$, with each firm described by two time-varying state variables $(M_{i,t}, X^*_{i,t})$. $M_{i,t}$ denotes the "installed base" of firm i, which are the share of consumers who have previously bought firm i's product. $X^*_{i,t}$ is firm i's marginal cost, which is unobserved to the econometrician, and is an unobserved state variable. There is learning by doing, in the sense that increases in the installed base will lower future marginal costs. In each period, each firm's choice variable $Y_{i,t}$ is its period t price, which affects the demand for its product in period t and thereby the future installed base, which in turn affects future production costs.

In the following, let $Y_t \equiv (Y_{1,t}, Y_{2,t})$, and similarly for M_t and X^*_t. Let $S_t \equiv (M_t, X^*_t)$ denote the payoff-relevant state variables of the game in period t. $S_{i,t} \equiv (M_{i,t}, X^*_{i,t})$, for $i = 1, 2$, denotes firm i's state variables. Each period, firms earn profits by selling their products to consumers who have not yet bought the product. The demand curve for firm i's product is

$$q_i(Y_t, M_t, \eta_{i,t})$$

which depends on the price and installed base of both firms' products. Firm i's demand also depends on $\eta_{i,t}$, a firm-specific demand shock. As in Aguirregabiria and Mira (2007) and Pesendorfer and Schmidt-Dengler (2008), we assume that $\eta_{i,t}$ is privately observed by each firm; that is, only firm 1, but not firm 2, observes $\eta_{1,t}$, making this a game of incomplete information. Furthermore, we assume that the demand shocks $\eta_{i,t}$ are i.i.d. across firm and periods, and distributed according to a distribution K which is common knowledge to both firms. The main role of the variable $\eta_{i,t}$ is to generate randomness in $Y_{i,t}$, even after conditioning on (M_t, X_t^*).

The period t profits of firm i can then be written:

$$\Pi_i(Y_t, S_t, \eta_{i,t}) = q_i(Y_t, M_t, \eta_{i,t}) * (Y_{i,t} - X_{i,t}^*)$$

where $Y_{i,t} - X_{i,t}^*$ is firm i's margin from each unit that it sells.

Installed base evolves according to the conditional distribution:

$$M_{i,t+1} \sim G(\cdot | M_{i,t}, Y_{i,t}). \tag{3}$$

One example is to model the incremental change $M_{i,t+1} - M_{i,t}$ as a log-normal random variable

$$\log(M_{i,t+1} - M_{i,t}) \sim q_i(Y_t, M_t, \eta_{i,t}) + \epsilon_{i,t}, \quad \epsilon_{i,t} \sim N(0, \sigma_\epsilon^2), \text{i.i.d.-}(i,t).$$

Marginal cost evolves according to the conditional distribution

$$X_{i,t+1}^* \sim H(\cdot | X_{i,t}^*, M_{i,t+1}). \tag{4}$$

One example is

$$X_{i,t+1}^* = X_{i,t}^* - N(\gamma(M_{i,t+1} - M_{i,t}), \sigma_k^2)$$

where γ and σ_k are unknown parameters. This encompasses learning-by-doing because the incremental reduction in marginal cost $(X_{i,t+1}^* - X_{i,t}^*)$ depends on the incremental increase in installed base $(M_{i,t+1} - M_{i,t})$.

In the dynamic Markov-perfect equilibrium, each firm's optimal pricing strategy will also be a function of the current S_t, and the current demand shock $\eta_{i,t}$:

$$Y_{i,t} = Y_i^*(S_t, \eta_{i,t}), \quad i = 1, 2 \tag{5}$$

where the strategy satisfies the equilibrium Bellman equation:

$$Y_i^*(S_t, \eta_{i,t}) = \text{argmax}_y E_{\eta_{-i,t}} \Big\{ \Pi_i(S_t, y, Y_{-i,t} = Y_{-i}^*(S_t, \eta_{-i,t})) + \beta E\left[V_i(S_{t+1}, \eta_{i,t+1}) | y, Y_{-i,t} = Y_{-i}^*(S_t, \eta_{-i,t})\right] \Big\} \tag{6}$$

subject to Eqs. (4) and (3). In the above equation, $V_i(S_t, \eta_{it})$ denotes the equilibrium value function for firm i, which is equal to the expected discounted future profits that firm i will make along the equilibrium path, starting at the current state (S_t, η_{it}). ■

Example 2 is a simplified version of the dynamic investment models estimated in the productivity literature. (See Ackerberg, Benkard, Berry, and Pakes (2007) for a detailed survey of this literature.) In this model, firms' state variables are $(M_{i,t}, X_{i,t}^*)$, where $M_{i,t}$ denotes firm i's capital stock, and $X_{i,t}^*$ denotes its productivity shock in period t. $Y_{i,t}$, firm i's choice variable, denotes new capital investment in period t.

Capital stock $M_{i,t}$ evolves deterministically, as a function of $(Y_{i,t-1}, M_{i,t-1})$:

$$M_{i,t} = (1 - \delta) \cdot M_{i,t-1} + Y_{i,t-1}. \tag{7}$$

The productivity shock is serially correlated, and evolves according to the conditional distribution

$$X_{i,t+1}^* \sim H(\cdot | X_{i,t}^*, M_{i,t}). \tag{8}$$

Each period, firms earn profits by selling their products. Let $q_i(p_{i,t}, p_{-i,t}, \eta_{i,t})$ denote the demand curve for firm i's product, which depends on the quality and prices of both firms' products. As in Example 1, $\eta_{i,t}$ denotes the privately observed demand shock for firm i in period t, which is distributed i.i.d. across firms and time periods.

The period t profits of firm i are:

$$q_i(p_{i,t}, p_{-i,t}, \eta_{i,t}) * (p_{i,t} - c_i(S_{i,t})) - K(Y_{i,t})$$

where $c_i(\cdot)$ is the marginal cost function for firm i (we assume constant marginal costs) and $K(Y_{it})$ is the investment cost function.

Following the literature, we assume that each firm's price in period t are determined by a static equilibrium, given the current values of the state variables S_t, and the firm-specific demand shock $\eta_{i,t}$. Let $p_i^*(S_t, \eta_{i,t})$ denote the static equilibrium prices for each firm in period t. By substituting in the equilibrium prices in firm's profit function, we obtain each firm's "reduced-form" expected profits:

$$\Pi_i(S_t, Y_t, \eta_{i,t}) = E_{\eta_{-i,t}} q_i\big(p_1^*(S_t, \eta_{1,t}), p_2^*(S_t, \eta_{2,t}), \eta_{i,t}\big) * \left[p_i^*(S_t, \eta_{i,t}) - c_i(S_{i,t})\right] - K(Y_{i,t}), \quad i = 1, 2$$

As in Example 1, the Markov equilibrium investment strategy for each firm just depends on the current state variables S_t, and the current shock $\eta_{i,t}$:

$$Y_t = Y_i^*(S_t, \eta_{it}), \quad i = 1, 2.$$

subject to the Bellman equation (6) and the transitions (7) and (8). ■

The substantial difference between examples 1 and 2 is that in example 2, the evolution of the observed state variable $M_{i,t}$ is deterministic, whereas in example 1 there is randomness in $M_{i,t}$ conditional on $(M_{i,t-1}, Y_{i,t-1})$ (i.e., compare Eqs. (3) and (7)). As we will see below, this has important implications for nonparametric identification.

Moreover, as illustrated in these two examples, for the first part of the paper we focus on games with continuous actions, so that Y_t are continuous variables. Later, we will consider the important alternative case of discrete-action games, where Y_t is discrete-valued.

3 Nonparametric identification

In this section, we present the assumptions for nonparametric identification in the dynamic game model. The assumption we make here are different than those in our earlier paper (Hu and Shum (2008)), and are geared specifically for the dynamic games literature, and motivated directly by existing applied work utilizing dynamic games. We assume that for each market j, $\left\{\left(W_{t+1}, X_{t+1}^*\right), \left(W_t, X_t^*\right), ..., \left(W_1, X_1^*\right)\right\}_j$ is an independent random draw from the identical distribution $f_{W_{t+1}, W_t, ..., W_1, X_{t+1}^*, X_t^*, ..., X_1^*}$. This rules out across-market effects and spillovers. For each market j, $\{W_1, \ldots, W_T\}_j$ is observed, for $T \geq 4$.

After presenting each assumption, we relate it to the examples in the previous section. Define $\Omega_{<t} = \left\{W_{t-1}, ..., W_1, X_{t-1}^*, ..., X_1^*\right\}$. We assume the dynamic process satisfies:

Assumption 1 *First-order Markov:*

$$f_{W_t,X_t^*|W_{t-1},X_{t-1}^*,\Omega_{<t-1}} = f_{W_t,X_t^*|W_{t-1},X_{t-1}^*}; \tag{9}$$

Remark: The first-order Markov assumption is satisfied along the Markov-equilibrium path of both examples given in the previous section. ■

Without loss of generality, we assume that $W_t = (Y_t, M_t) \in \mathbb{R}^2$. We assume

Assumption 2

$$\begin{array}{ll} \textit{(i)} & f_{Y_t|M_t,X_t^*,Y_{t-1},M_{t-1},X_{t-1}^*} = f_{Y_t|M_t,X_t^*}, \\ \textit{(ii)} & f_{X_t^*|M_t,Y_{t-1},M_{t-1},X_{t-1}^*} = f_{X_t^*|M_t,M_{t-1},X_{t-1}^*}. \end{array}$$

Assumption 2(i) is motivated completely by the state-contingent aspect of the optimal policy function in dynamic optimization models. It turns out that this assumption is stronger than necessary for our identification, but it allows us to achieve identification only using three periods of data. Assumption 2(ii) implies that X_t^* is independent of Y_{t-1} conditional on M_t, M_{t-1} and X_{t-1}^*. This is consistent with the setup above.

Remarks: Assumption 2 is satisfied in both examples 1 and 2. ■

The conditional independence assumptions 1-2 imply that the Markov transition density (1) can be factored into

$$\begin{aligned} f_{W_t,X_t^*|W_{t-1},X_{t-1}^*} &= f_{Y_t,M_t,X_t^*|Y_{t-1},M_{t-1},X_{t-1}^*} \\ &= f_{Y_t|M_t,X_t^*} \cdot f_{X_t^*|M_t,M_{t-1},X_{t-1}^*} \cdot f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*}. \end{aligned} \tag{10}$$

In the identification procedure, we will identify these three components of $f_{W_t,X_t^*|W_{t-1},X_{t-1}^*}$ in turn.

Next, we restrict attention to stationary equilibria in the dynamic game, which is natural given our focus on Markov equilibria. In stationary equilibria, the Markov transition density $f_{W_t,X_t^*|W_{t-1},X_{t-1}^*}$ is time-invariant.

Assumption 3 *Stationarity of Markov kernel:*

$$f_{W_t,X_t^*|W_{t-1},X_{t-1}^*} = f_{W_2,X_2^*|W_1,X_1^*}.$$

For simplicity, we assume that $Y_{i,t},\ M_t,\ X_{i,t}^* \in \{1,2,...,J\}$. Consider the joint density of $\{Y_t, M_t, Y_{t-1}, M_{t-1}, Y_{t-2}\}$. We show in the Appendix, that Assumptions 1-2 imply that

$$\begin{aligned} & f_{Y_t,M_t,Y_{t-1}|M_{t-1},Y_{t-2}} \\ = & \sum_{x_{t-1}^*} f_{Y_t|M_t,M_{t-1},X_{t-1}^*} f_{M_t,Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*|M_{t-1},Y_{t-2}} \end{aligned} \tag{11}$$

where the final line follows from Assumptions 1-2.

In order to identify the unknown densities on the right hand side, we use the identification strategy for the nonclassical measurement error models in Hu (2008). His results imply that two measurements and a dependent variable of a latent explanatory variable are enough to achieve identification. For fixed values of (M_t, M_{t-1}), we see that (Y_t, Y_{t-1}, Y_{t-2}) enter equation (11) separately in, respectively, the first, second, and third terms. This implies that we can use (Y_t, Y_{t-2}) as the two measurements and Y_{t-1} as the dependent variable of the latent variable X_{t-1}^*.

We abuse the notation Y_t and define

$$Y_t = G(Y_{1,t}, Y_{2,t}) \equiv \begin{cases} 1 & \text{if} \quad (Y_{1,t}, Y_{2,t}) = (1,1) \\ 2 & \text{if} \quad (Y_{1,t}, Y_{2,t}) = (1,2) \\ ... & \qquad\qquad ... \\ J^2 & \text{if} \quad (Y_{1,t}, Y_{2,t}) = (J,J) \end{cases},$$

where the one-to-one function G maps a vector of discrete variables to a scalar discrete variable. Similarly, we may also redefine $X_t^* = G(X_{1,t}^*, X_{2,t}^*)$. Furthermore, we define the matrix $\mathbf{F}_{Y_t,m_t,y_{t-1}|m_{t-1},Y_{t-2}}$ for any given (m_t, y_{t-1}, m_{t-1}) in the support of (M_t, Y_{t-1}, M_{t-1})

and $i,j,k \in \mathcal{S} \equiv \left\{1, 2..., J^2\right\}$

$$
\begin{aligned}
\mathbf{F}_{Y_t,m_t,y_{t-1}|m_{t-1},Y_{t-2}} &= \left[f_{Y_t,M_t,Y_{t-1}|M_{t-1},Y_{t-2}}\left(i, m_t, y_{t-1}|m_{t-1}, j\right)\right]_{i,j}, \\
\mathbf{F}_{Y_t|m_t,m_{t-1},X_{t-1}^*} &= \left[f_{Y_t|M_t,M_{t-1},X_{t-1}^*}(i|m_t, m_{t-1}, k)\right]_{i,k}, \\
\mathbf{D}_{y_{t-1}|m_t,m_{t-1},X_{t-1}^*} &= diag\left\{\left[f_{Y_{t-1}|M_t,M_{t-1},X_{t-1}^*}(y_{t-1}|m_t, m_{t-1}, k)\right]_k\right\}, \\
\mathbf{D}_{m_t|m_{t-1},X_{t-1}^*} &= diag\left\{\left[f_{M_t|M_{t-1},X_{t-1}^*}(m_t|m_{t-1}, k)\right]_k\right\}, \\
\mathbf{F}_{X_{t-1}^*|m_{t-1},Y_{t-2}} &= \left[f_{X_{t-1}^*|M_{t-1},Y_{t-2}}(k|m_{t-1}, j)\right]_{k,j},
\end{aligned}
$$

where $diag\left\{V\right\}$ generates a diagonal matrix with diagonal entries equal to the corresponding ones in the vector V. As shown in the Appendix, equation (11) can be written in matrix notation as (for fixed (m_t, y_{t-1}, m_{t-1})):

$$
\mathbf{F}_{Y_t,m_t,y_{t-1}|m_{t-1},Y_{t-2}} = \mathbf{F}_{Y_t|m_t,m_{t-1},X_{t-1}^*}\mathbf{D}_{y_{t-1}|m_t,m_{t-1},X_{t-1}^*}\mathbf{D}_{m_t|m_{t-1},X_{t-1}^*}\mathbf{F}_{X_{t-1}^*|m_{t-1},Y_{t-2}}. \tag{12}
$$

Similarly, integrating our y_{t-1} in equation 11 leads to for any given (m_t, m_{t-1})

$$
\mathbf{F}_{Y_t,m_t|m_{t-1},Y_{t-2}} = \mathbf{F}_{Y_t|m_t,m_{t-1},X_{t-1}^*}\mathbf{D}_{m_t|m_{t-1},X_{t-1}^*}\mathbf{F}_{X_{t-1}^*|m_{t-1},Y_{t-2}}, \tag{13}
$$

where

$$
\mathbf{F}_{Y_t,m_t|m_{t-1},Y_{t-2}} = \left[f_{Y_t,M_t|M_{t-1},Y_{t-2}}\left(i, m_t|m_{t-1}, j\right)\right]_{i,j}.
$$

The identification of a matrix, e.g., $\mathbf{F}_{Y_t|m_t,m_{t-1},X_{t-1}^*}$, is equivalent to that of its corresponding density, e.g., $f_{Y_t|M_t,M_{t-1},X_{t-1}^*}$. Identification of $\mathbf{F}_{Y_t|m_t,m_{t-1},X_{t-1}^*}$ from the observed $\mathbf{F}_{Y_t,m_t,y_{t-1}|m_{t-1},Y_{t-2}}$ requires

Assumption 4 *For any (m_t, m_{t-1}), there exists a $y_{t-1} \in \mathcal{S}$ such that $L_{Y_t,m_t,y_{t-1}|m_{t-1},Y_{t-2}}$ is invertible.*

Assumption 4(ii) rules out cases where the support of X_{t-1}^* is larger than that of Y_t. Hence, in this section, we are restricting attention to the case where Y_t and X_{t-1}^* have the same support.

Remark: This assumption implies that all the unknown matrices on the right hand side are invertible. In particular, all the diagonal entries in $\mathbf{D}_{y_{t-1}|m_t,m_{t-1},X_{t-1}^*}$ and $\mathbf{D}_{m_t|m_{t-1},X_{t-1}^*}$ are nonzero. Furthermore, this assumption is directly testable using the sample. ■

As in Hu (2008), if the latter matrix relation can be inverted (which is ensured by Assumption 4), we can combine Eqs. (12) and (13) to get

$$\begin{aligned}
& \mathbf{F}_{Y_t,m_t,y_{t-1}|m_{t-1},Y_{t-2}}\mathbf{F}^{-1}_{Y_t,m_t,y_{t-1}|m_{t-1},Y_{t-2}} \\
= \; & \mathbf{F}_{Y_t|m_t,m_{t-1},X_{t-1}^*}\cdot \mathbf{D}_{y_{t-1}|m_t,m_{t-1},X_{t-1}^*}\cdot \mathbf{F}^{-1}_{Y_t|m_t,m_{t-1},X_{t-1}^*}.
\end{aligned} \tag{14}$$

This representation shows that an eigenvalue-eigenfunction decomposition of the observed matrix $\mathbf{F}_{Y_t,m_t,y_{t-1}|m_{t-1},Y_{t-2}}\mathbf{F}^{-1}_{Y_t,m_t,y_{t-1}|m_{t-1},Y_{t-2}}$ yields the unknown density functions $f_{Y_t|m_t,m_{t-1},X_{t-1}^*}$ as the eigenfunctions and $f_{y_{t-1}|m_t,m_{t-1},X_{t-1}^*}$ as the eigenvalues.

The following assumption ensures the uniqueness of this decomposition, and restricts the choice of the $\omega(\cdot)$ function.

Assumption 5 *For any (m_t, m_{t-1}), there exists a $y_{t-1} \in \mathcal{S}$ such that for $j \neq k \in \mathcal{S}$*

$$f_{Y_{t-1}|M_t,M_{t-1},X_{t-1}^*}(y_{t-1}|m_t,m_{t-1},j) \neq f_{Y_{t-1}|M_t,M_{t-1},X_{t-1}^*}(y_{t-1}|m_t,m_{t-1},k).$$

Assumption 5 implies that the latent variable does change the distribution of Y_{t-1} given M_t in the two periods. Notice that assumption 4 guarantees that $f_{y_{t-1}|m_t,m_{t-1},X_{t-1}^*} \neq 0$.

Remark: Assumption 5 requires that the conditional density $f_{Y_{t-1}|M_t,M_{t-1},X_{t-1}^*}(y_{t-1}|m_t,m_{t-1},x_{t-1}^*)$ varies in X_{t-1}^* given any fixed (m_t, m_{t-1}), so that the "eigenvalues" in the decomposition (14) are distinctive. For example 1, given the preceding discussion, assumption 5 should hold. For example 2, the capital stock M_t evolves deterministically, so that $f_{Y_{t-1}|M_t,M_{t-1},X_{t-1}^*}(y_{t-1}|m_t,m_{t-1},x_{t-1}^*) = I\left(y_{t-1} = m_t - (1-\delta)m_{t-1}\right)$. Since this does not change with x_{t-1}^* for any fixed (m_t, m_{t-1}), Therefore, assumption 5 fails. ■

Remark (Deterministic choices): In some models, the choice variable Y_{it} is a deterministic function of the current state variables, i.e.,

$$Y_{i,t-1} = g_i(M_{t-1}, X_{t-1}^*), \quad i = 1, 2. \tag{15}$$

In examples 1 and 2, this would be the case if we eliminated the privately-observed demand shocks η_{1t} and η_{2t}. Assumption 5 becomes

$$f_{Y_{t-1}|M_{t-1},X^*_{t-1}}(y_{t-1}|m_{t-1},j) \neq f_{Y_{t-1}|M_{t-1},X^*_{t-1}}(y_{t-1}|m_{t-1},k).$$

■

Remark: Notice that in the decomposition (14), y_{t-1} only appears in the eigenvalues. Therefore, if there are several values y_{t-1} which satisfy Assumption (5), the decompositions (14) using these different y_{t-1}'s should yield the same eigenfunctions. Hence, depending on the specific model, it may be possible to use this feature as a general specification check for Assumptions (1) and (2). We do not explore this possibility here. ■

Under the foregoing assumptions, the density $Y_t, m_t, y_{t-1}|m_{t-1}, Y_{t-2}$ can form a unique eigenvalue-eigenvector decomposition. In this decomposition, the eigenfunction corresponds to the density $f_{Y_t|m_t,m_{t-1},X^*_{t-1}}(\cdot|m_t,m_{t-1},x^*_{t-1})$ which can be written as

$$f_{Y_t|m_t,m_{t-1},X^*_{t-1}}(\cdot|m_t,m_{t-1},x^*_{t-1}) = f_{Y_{1,t},Y_{2,t}|m_t,m_{t-1},X^*_{1,t-1},X^*_{2,t-1}}(\cdot,\cdot|m_t,m_{t-1},x^*_{1,t-1},x^*_{2,t-1}). \tag{16}$$

The eigenvalue-eigenfunction decomposition only identifies this eigenfunction up to some arbitrary ordering of the $(x^*_{1,t-1}, x^*_{2,t-1})$ argument. Hence, in order to pin down the right ordering of x^*_{t-1}, an additional ordering assumption is required. In our earlier paper (Hu and Shum (2008)), where x^*_t was scalar-valued, a monotonicity assumption sufficed to pin down the ordering of x^*_t. However, in dynamic games, x^*_{t-1} is multivariate, so that monotonicity is no longer well-defined.

Consider the marginal density

$$f_{Y_{i,t}|m_t,m_{t-1},X^*_{1,t-1},X^*_{2,t-1}}(\cdot|m_t,m_{t-1},x^*_{1,t-1},x^*_{2,t-1}),$$

which can be computed from Eq. (16) above. We make the following ordering assumption:

Assumption 6 *for any given (m_t, m_{t-1}) and $j \neq k \in \mathcal{S}$*

$$f_{Y_t|m_t,m_{t-1},X^*_{t-1}}(k|m_t,m_{t-1},k) > f_{Y_t|m_t,m_{t-1},X^*_{t-1}}(j|m_t,m_{t-1},k).$$

Remark: With this assumption, the mode of $f_{Y_{1,t},Y_{2,t}|m_t,m_{t-1},X^*_{1,t-1},X^*_{2,t-1}}(\cdot,\cdot|m_t,m_{t-1},j,k)$ is (j,k). Therefore, the value of the latent variable $x^*_{1,t-1},x^*_{2,t-1}$ can be identified from the eigenvectors. In other words, the "pattern" of the latent marginal cost $\left(x^*_{1,t-1},x^*_{2,t-1}\right)$ is revealed at the mode of the price distribution of $(Y_{1,t},Y_{2,t})$. Both (i) and (ii) should be confirmed on a model-by-model basis, but is not unreasonable given the interpretation of $Y_{i,t}$ as a price and $X^*_{1,t}$ as a marginal cost variable. ■

From the eigenvalue-eigenvector decomposition in Eq. (14), Hu (2008) implies that we can identify all the unknown matrices $\mathbf{F}_{Y_t|m_t,m_{t-1},X^*_{t-1}}$, $\mathbf{D}_{y_{t-1}|m_t,m_{t-1},X^*_{t-1}}$, $\mathbf{D}_{m_t|m_{t-1},X^*_{t-1}}$, and $\mathbf{F}_{X^*_{t-1}|m_{t-1},Y_{t-2}}$ for any (m_t,y_{t-1},m_{t-1}) and their corresponding densities$f_{Y_t|M_t,M_{t-1},X^*_{t-1}}$, $f_{Y_{t-1}|M_t,M_{t-1},X^*_{t-1}}$, $f_{M_t|M_{t-1},X^*_{t-1}}$, and $f_{X^*_{t-1}|M_{t-1},Y_{t-2}}$. That implies we can identify $f_{M_t,Y_{t-1}|M_{t-1},X^*_{t-1}}$ as

$$f_{M_t,Y_{t-1}|M_{t-1},X^*_{t-1}} = f_{Y_{t-1}|M_t,M_{t-1},X^*_{t-1}} f_{M_t|M_{t-1},X^*_{t-1}}.$$

From the factorization

$$f_{M_t,Y_{t-1}|M_{t-1},X^*_{t-1}} = f_{M_t|Y_{t-1},M_{t-1},X^*_{t-1}} \cdot f_{Y_{t-1}|M_{t-1},X^*_{t-1}}$$

we can recover $f_{M_t|Y_{t-1},M_{t-1},X^*_{t-1}}$ and $f_{Y_{t-1}|M_{t-1},X^*_{t-1}}$. Given stationarity, the latter density is identical to $f_{Y_t|M_t,X^*_t}$, so that from $f_{M_t,Y_{t-1}|M_{t-1},X^*_{t-1}}$ we have recovered the first two components of $f_{W_t,X^*_t|W_{t-1},X^*_{t-1}}$ in Eq. (10).

All that remains now is to identify the third component $f_{X^*_t|M_t,M_{t-1},X^*_{t-1}}$. To obtain this, note that the following matrix relation holds:

$$\mathbf{F}_{Y_t|m_t,m_{t-1},X^*_{t-1}} = \mathbf{F}_{Y_t|m_t,X^*_t}\mathbf{F}_{X^*_t|m_t,m_{t-1},X^*_{t-1}}$$

for given (m_t,m_{t-1}), and where for i,l,$k \in \mathcal{S}$

$$\begin{aligned}
\mathbf{F}_{X^*_t|m_t,m_{t-1},X^*_{t-1}} &= \left[f_{X^*_t|M_t,M_{t-1},X^*_{t-1}}(l|m_t,m_{t-1},k)\right]_{l,k} \\
\mathbf{F}_{Y_t|m_t,X^*_t} &= \left[f_{Y_t|M_t,X^*_t}(i|m_t,l)\right]_{i,l}.
\end{aligned}$$

The invertibility of $\mathbf{F}_{Y_t|m_t,m_{t-1},X^*_{t-1}}$ implies that of $\mathbf{F}_{Y_t|m_t,X^*_t}$. Therefore, the final component in Eq. (10) can be recovered as:

$$\mathbf{F}_{X^*_t|m_t,m_{t-1},X^*_{t-1}} = \mathbf{F}^{-1}_{Y_t|m_t,X^*_t}\mathbf{F}_{Y_t|m_t,m_{t-1},X^*_{t-1}} \tag{17}$$

where both terms on the right-hand-side have already been identified in previous steps.

Finally, we summarize the identification results as follows:

Theorem 1 *(Stationary case) Under the assumptions 1, 2, 3, 4, 5, and 6, the density* $f_{W_t,W_{t-1},W_{t-2}}$*, for any* $t \in \{3, \ldots T\}$*, uniquely determines the time-invariant Markov equilibrium transition density* $f_{W_2,X_2^*|W_1,X_1^*}$.

Proof. See the appendix. ∎

This theorem implies that we may identify the Markov kernel density with three periods of data.

Without stationarity, the desired density $f_{Y_t|M_t,X_t^*}$ is not the same as $f_{Y_{t-1}|M_{t-1},X_{t-1}^*}$, which can be recovered from the three observations $f_{W_t,W_{t-1},W_{t-2}}$. However, in this case, we can repeat the whole foregoing argument for the three observations $f_{W_{t+1},W_t,W_{t-1}}$ to identify $f_{Y_t|M_t,X_t^*}$. Hence, the following corollary is immediate:

Corollary 1 *(Nonstationary case) Under the assumptions 1, 2, 4, 5, and 6, the density* $f_{W_{t+1},W_t,W_{t-1},W_{t-2}}$ *uniquely determines the time-varying Markov equilibrium transition density* $f_{W_t,X_t^*|W_{t-1},X_{t-1}^*}$*, for every period* $t \in \{3, \ldots T-1\}$.

4 Extensions

4.1 Alternatives to Assumption 2(ii)

In this section, we consider alternative conditions of assumption 2(ii). Assumption 2(ii) implies that X_t^* is independent of Y_{t-1} conditional on M_t, M_{t-1} and X_{t-1}^*. There are other alternative "limited feedback" assumptions, which may be suitable for different empirical settings. Assumptions 1 and 2(i) imply

$$\begin{aligned}
& f_{W_{t+1},W_t,W_{t-1},W_{t-2}} \\
= \; & f_{Y_{t+1},M_{t+1},Y_t,M_t,Y_{t-1},M_{t-1},Y_{t-2},M_{t-2}} \\
= \; & \int\int \Big[f_{Y_{t+1},M_{t+1}|Y_t,M_t,X_t^*} f_{Y_t|M_t,X_t^*} f_{X_t^*,M_t|Y_{t-1},M_{t-1},X_{t-1}^*} \cdot \\
& f_{Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*,M_{t-1},Y_{t-2},M_{t-2}} \Big] dx_t^* dx_{t-1}^*.
\end{aligned}$$

Assumption 2(ii) implies that the state transition density satisfies

$$f_{X_t^*,M_t|Y_{t-1},M_{t-1},X_{t-1}^*} = f_{X_t^*|M_t,M_{t-1},X_{t-1}^*} f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*}.$$

Alternative "limited feedback" assumptions may be imposed on the density $f_{X_t^*,M_t|Y_{t-1},M_{t-1},X_{t-1}^*}$. One alternative to assumption 2(ii) is

$$f_{X_t^*,M_t|Y_{t-1},M_{t-1},X_{t-1}^*} = f_{X_t^*|M_t,Y_{t-1},X_{t-1}^*} f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*}, \tag{18}$$

which implies that M_{t-1} does not have a direct effect on X_t^* conditional on M_t, Y_{t-1}, and X_{t-1}^*. A second alternative is

$$f_{X_t^*,M_t|Y_{t-1},M_{t-1},X_{t-1}^*} = f_{M_t|X_t^*,Y_{t-1},M_{t-1}} f_{X_t^*|Y_{t-1},M_{t-1},X_{t-1}^*}. \tag{19}$$

which is the "limited feedback" assumption used in our earlier study (Hu and Shum (2008)) of identification on single-agent dynamic optimization problems. Both alternatives (18) and (19) can be handled using identification arguments similar to the one in Hu and Shum (2008).

A third alternative to assumption 2(ii) is

$$f_{X_t^*,M_t|Y_{t-1},M_{t-1},X_{t-1}^*} = f_{X_t^*|M_t,Y_{t-1},M_{t-1},X_{t-1}^*} f_{M_t|M_{t-1},X_{t-1}^*}. \tag{20}$$

This alternative can be handled in an identification framework similar to the one used in this paper.

5 Conclusions

In this paper, we show several results regarding nonparametric identification in a general class of Markov dynamic games, including many models in the Ericson and Pakes (1995) and Pakes and McGuire (1994) framework. We show that only three observations $W_t, \ldots, W_{t-2}$ are required to identify $W_t, X_t^*|W_{t-1}, X_{t-1}^*$ in the stationary case, when Y_t is a continuous choice variable. If Y_t is a discrete choice variable (while X_t^* is continuous), then four observations are required for identification.

In ongoing work, we are working on developing estimation procedures for dynamic games which utilize these identification results.

Proof. (theorem 1) First, assumptions 1-2 imply that the density of interest becomes

$$\begin{aligned} f_{W_t,X_t^*|W_{t-1},X_{t-1}^*} &= f_{Y_t,M_t,X_t^*|Y_{t-1},M_{t-1},X_{t-1}^*} \\ &= f_{Y_t|M_t,X_t^*,Y_{t-1},M_{t-1},X_{t-1}^*} f_{X_t^*|M_t,Y_{t-1},M_{t-1},X_{t-1}^*} f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*} \\ &= f_{Y_t|M_t,X_t^*} f_{X_t^*|M_t,M_{t-1},X_{t-1}^*} f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*}. \end{aligned} \tag{21}$$

We consider the observed density $f_{W_t,W_{t-1},W_{t-2}}$. One can show that assumptions 1 and 2(i) imply

$$\begin{aligned} & f_{W_t,W_{t-1},W_{t-2}} \\ = & \sum_{x_t^*} \sum_{x_{t-1}^*} f_{W_t,X_t^*|W_{t-1},W_{t-2},X_{t-1}^*} f_{W_{t-1},W_{t-2},X_{t-1}^*} \\ = & \sum_{x_t^*} \sum_{x_{t-1}^*} f_{Y_t|M_t,X_t^*} f_{X_t^*|M_t,Y_{t-1},M_{t-1},X_{t-1}^*} f_{M_t|Y_{t-1},M_{t-1},X_{t-1}^*} f_{Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*,M_{t-1},Y_{t-2},M_{t-2}} \\ = & \sum_{x_t^*} \sum_{x_{t-1}^*} f_{Y_t|M_t,X_t^*} f_{X_t^*|M_t,Y_{t-1},M_{t-1},X_{t-1}^*} f_{M_t,Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*,M_{t-1},Y_{t-2},M_{t-2}}. \end{aligned}$$

After integrating out M_{t-2}, assumption 2(ii) then implies

$$\begin{aligned} & f_{Y_t,M_t,Y_{t-1},M_{t-1},Y_{t-2}} \\ = & \sum_{x_{t-1}^*} \left(\sum_{x_t^*} f_{Y_t|M_t,X_t^*} f_{X_t^*|M_t,M_{t-1},X_{t-1}^*} \right) f_{M_t,Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*,M_{t-1},Y_{t-2}} \end{aligned}$$

The expression in the parenthesis can be simplified as $f_{Y_t|M_t,M_{t-1},X_{t-1}^*}$. We then have

$$\begin{aligned} & f_{Y_t,M_t,Y_{t-1}|M_{t-1},Y_{t-2}} \\ = & \sum_{x_{t-1}^*} f_{Y_t|M_t,M_{t-1},X_{t-1}^*} f_{M_t,Y_{t-1}|M_{t-1},X_{t-1}^*} f_{X_{t-1}^*|M_{t-1},Y_{t-2}}. \end{aligned} \tag{22}$$

Straightforward algebra shows that this equation is equivalent to

$$\mathbf{F}_{Y_t,m_t,y_{t-1}|m_{t-1},Y_{t-2}} = \mathbf{F}_{Y_t|m_t,m_{t-1},X_{t-1}^*} \mathbf{D}_{y_{t-1}|m_t,m_{t-1},X_{t-1}^*} \mathbf{D}_{m_t|m_{t-1},X_{t-1}^*} \mathbf{F}_{X_{t-1}^*|m_{t-1},Y_{t-2}}. \tag{23}$$

for any given (m_t, y_{t-1}, m_{t-1}). The identification results then follow from theorem 1 in Hu (2008). ∎

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