Identification of First-Price Auctions with Non-separable Unobserved Heterogeneity*

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Abstract

We propose a novel methodology for identification of first-price auctions, when bidders' private valuations are independent conditional on one-dimensional unobserved heterogeneity. We extend the existing literature (Li and Vuong (1998), Krasnokutskaya (2011)) by allowing the unobserved heterogeneity to be nonseparable from bidders' valuations. Our central identifying assumption is that the distribution of bidder values is increasing in the state. When the state-space is finite, such monotonicity implies the completeness conditions needed for identification. When the state-space is continuous, we also provide some new sufficient conditions which ensure that completeness holds. Further, we extend our approach to the conditionally independent private values model of Li, Perrigne, and Vuong (2000), as well as to unobserved heterogeneity settings in which the implicit reserve price or the cost of bidding varies across auctions.

1 Introduction

This paper considers the problem of identification in first-price auctions in which bidders have independent private values conditional on an unobserved one-dimensional state Y.

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Given the joint distribution of bids, $(B_1, ..., B_n)$, what can be inferred about the joint distribution of bidder values and the state, $(V_1, ..., V_n, Y)$? In many empirical applications, such a state variable Y captures an auction-specific characteristic commonly observed by the bidders but unobserved by the econometrician. The resulting model is one of *Independent Private Values with Unobserved Heterogeneity ("UH model").*

The existing literature has mainly focused on the convolution case, in which the unobserved heterogeneity has either an additive or multiplicative effect on bidder values, e.g, $V_i = S_i Y$ with independent signals S_i that are independent of Y (Li and Vuong (1998), Krasnokutskaya (2011)). Our approach to identify $(V_1, ..., V_n, Y)$ differs in that we rely instead on the weaker assumption that the distribution of bidder values is *monotone* in the state, in the sense of first-order stochastic dominance ("FOSD").

This additional generality in the relationship between V and Y comes at some cost. Our identification approach relies on recent results in the econometric literature on nonlinear measurement error (Hu (2008) and Hu and Schennach (2008)). At least *three* bidders per auction are required to apply these results, whereas previous approaches have required only two bidders per auction. In addition, for our results to hold, the distribution of equilibrium bids conditional on the state must satisfy certain completeness conditions. However, when the state-space is finite, completeness follows immediately from our monotonicity assumption on the underlying distribution of bidder values. When the state-space is continuous, sufficient primitive conditions are more difficult to find, but we provide new sufficient conditions that shed light on the restrictiveness of the completeness assumption.

Most of the paper focuses on a setting in which the state can be interpreted as product quality that is observed by the bidders but not the econometrician. In particular, higher states correspond to FOSD-higher conditional distributions of bidder values. While the distribution of equilibrium bids need not be FOSD-increasing in the state,[1] the *maximum of the equilibrium bid support* is increasing in the state. Such "monotonicity of the maximum" is enough to satisfy the monotonicity condition required by our approach. Further, when the state-space is finite, monotonicity of the maximum is also enough to satisfy the completeness conditions required by our approach.

In Section 3, we consider the *Conditionally Independent Private Value ("CIPV")* model, the identification and estimation of which has been analyzed in Li, Perrigne, and Vuong

[1] Lebrun (1998) shows that the distribution of equilibrium bids is increasing in the distribution of bidder values, with respect to a *stricter* stochastic order than first-order stochastic dominance, where $G \succ H$ iff $\frac{d}{dv}\frac{G}{H}(v) > 0$ for all v.

(2000, 2002). This model is behaviorally distinct from but statistically similar to the UH model. The key difference between the CIPV model and the UH model is that, in the CIPV model, bidders also do not observe the state. As we shall see, this distinction makes the CIPV model *easier* to analyze. In particular, we can provide sufficient primitive conditions on the joint distribution of bidder values and the state for completeness to be satisfied, even in the more difficult case of a continuous state-space.

To illustrate the breadth of applicability of our identification approach, Section 4 considers two settings in which the state induces unobserved heterogeneity in the distribution of bids *without* affecting the underlying distribution of bidder values. First, suppose that the seller's (implicit) reserve price is known to the bidders but not observed by the econometrician. The maximum of the equilibrium bid support is higher when bidders are faced with a higher reserve. Thus, the central monotonicity assumption of our approach is satisfied, and we identify the joint distribution of bidder values and the reserve. Second and similarly, suppose that bidding is costly, but the cost of bidding is not observed by the econometrician. The maximum of the equilibrium bid support is higher when bidders are faced with a lower cost of bidding, again satisfying our monotonicity requirement when the space of possible costs is endowed with the reverse order. In this case, we identify the joint distribution of bidder values and the bidding cost.

Related literature. Our paper builds on the pioneering work of Li and Vuong (1998), Li, Perrigne, and Vuong (2000, 2002), and Krasnokutskaya (2011), who applied results from the literature on classical measurement error to identify CIPV models and UH models in which the state has a separable effect on bidder values.[2,3] We relax this separability assumption by applying non-classical measurement error results in Hu (2008) and Hu and Schennach (2008), though at the cost of requiring three bids per auction rather than two.

Another closely related paper is An, Hu, and Shum (2010), in which unobserved heterogeneity takes the form of an *unobserved number of potential bidders.* An, Hu, and Shum (2010)'s identification approach exploits the fact that the number of observed bidders is always less than or equal to the number of potential bidders. By contrast, our approach exploits the fact that some location of the distribution of equilibrium bids is increasing in the unobserved state. Thus, while their approach is specific to their particular application,

[2]These papers also developed important results on estimation. By contrast, we only address the problem of identification.

[3]Krasnokutskaya (2009) has recently extended her approach to a setting in which a *multi-dimensional* state has a separable effect on bidder values.

our approach can be adapted to a wide variety of auction settings that exhibit monotonicity of bids in the state. Indeed, our approach can even be adapted to identify An, Hu, and Shum (2010)'s model, since the distribution of equilibrium bids is increasing in the number of potential bidders N^*.[4] Further, An, Hu, and Shum focuses exclusively on the UH model with a discrete state-space, while we (i) allow for a discrete or continuous state-space in the UH model and (ii) extend our analysis to the CIPV model.

Several other papers in the recent literature address identification when there is an unobserved state. Most similar in spirit is d'Haultfoeille and Fevrier (2008), who identify a common value model with conditionally independent signals, when there are at least three bidders and the support of bidders' signals is strictly increasing in the underlying common value. Roberts (2009) identifies a model with independent private values and unobserved heterogeneity given two bidders plus a reserve price that serves as an instrument for the unobserved heterogeneity. Aradillas-Lopez, Gandhi, and Quint (2010) partially identify a model with unobserved heterogeneity and correlated private values given data generated in an ascending auction. Our identification approach, which is based on measurement error results, is quite distinct from all of these papers.

The rest of the paper is organized as follows. Section 2 introduces and analyzes our main model of unobserved heterogeneity in first-price auctions with independent private values ("UH model"). Sufficient conditions for monotonicity and completeness are discussed in detail in Sections 2.1-2.2. Section 3 extends our approach to a setting with conditionally independent private values ("CIPV model"). Two other UH applications, to settings with an unobserved implicit reserve price or an unobserved cost of bidding, are considered in Section 4. Section 5 concludes.

2 Independent private values with unobserved heterogeneity

Model. $n \geq 3$ symmetric risk-neutral bidders participate in a first-price auction with zero reserve price. Conditional on an auction-specific "state" $Y \in \mathcal{Y} \subset \mathbb{R}$, bidders' private values $(V_1, ..., V_n)$ are independent conditional on Y and bounded with support $[0, \overline{v}]$, and the conditional pdf $f(v_i|Y = y)$ is continuously differentiable in v_i and bounded away from both zero and infinity on its support. (These assumptions are sufficient to imply existence

[4] An, Hu, and Shum assume that the distribution of bidder values does not depend on N^*, and that $N^* \geq 3$ with positive probability. As long as bidder identities are observed, one can then apply our identification results to the joint distribution of three bidders' bids – including when they each fail to bid.

of a unique bidding equilibrium, which is in symmetric strategies.) Let $f(v_i, y)$ denote the joint density of (V_i, Y).[5]

The state Y is common knowledge among the bidders prior to the bidding, but the econometrician knows nothing about the state, not even the distribution from which it is drawn. Y constitutes a source of unobserved heterogeneity, since equilibrium bidding strategies vary with the state. Our central identifying assumption is a monotonicity property on model primitives, that the distribution of bidder values is increasing in the state.

Monotonicity Assumption. $V_i|(Y = y)$ is strictly increasing in y with respect to first-order stochastic dominance ("FOSD-increasing"). That is, $y' > y$ implies $F(v_i|y') \leq F(v_i|y)$ for all v_i, with strict inequality at some v_i.

Proposition 1 (Monotonicity of the maximum) *Let $\overline{b}(y) = \max \text{supp}(B_i|Y = y)$ be the maximum of the equilibrium bid support, conditional on the state. When the Monotonicity Assumption is satisfied, $\overline{b}(y)$ is strictly increasing in y.*

Proof. By the Envelope Theorem, a bidder with value v_i earns interim expected payoff equal to $\int_0^{v_i} F(v|y)^{n-1}dv$, where $F(v|y)^{n-1}$ is each bidder's conditional probability of winning given value v. Since a bidder with the highest possible value $\overline{v}$ always wins, this means that such a bidder's equilibrium bid equals

$$\overline{b}(y) = b(\overline{v}|y) = \overline{v} - \int_0^{\overline{v}} F(v|y)^{n-1}dv.$$

Suppose $y' > y$. By the Monotonicity Assumption, $F(v|y') \leq F(v|y)$ for all v, with strict inequality for a positive measure of values. Thus, $\overline{b}(y') > \overline{b}(y)$. ■

Let $\mathcal{B} = \mathbb{R}_+$ be the space of permissible bids, $G(b_i|Y = y)$ and $g(b_i|Y = y)$ the cdf and pdf of each bidder's equilibrium bid conditional on the state, and $g(b_i, y)$ the joint density of (B_i, Y). Note that, since bidder values are i.i.d. conditional on Y, so are bids.

We assume that the joint density $g(b_i, y)$ is bounded, as are all conditional and marginal densities. Further, we assume that the state-space has a minimal element, $\min Y = \underline{y}$. Since the equilibrium bid support is $[0, \overline{b}(y)]$ in each state y, Proposition 1 then implies that $[0, \overline{b}(\underline{y})]$ is part of the equilibrium bid support in all states.

[5]Throughout the paper, random variables are capitalized while realizations are in lower case.

Discussion of the model. *Symmetry.* All of our analysis can be easily generalized to allow for asymmetric bidders, as long as one is willing to assume (or able to prove) that (i) in the CIPV model, there is a unique bidding equilibrium and (ii) in the UH model, the maximum of each bidder's equilibrium bid support (or some other location of the distribution of bids, such as the mean) is higher in higher states.[6] When bidders are symmetric, these uniqueness and monotonicity properties are well-established. Namely, (i) McAdams (2007) proves uniqueness in the CIPV model when bidders are symmetric, while (ii) Proposition 1 establishes monotonicity of the maximum when bidders are symmetric.

Three bidders. If there are only two bidders, our analysis still applies if an appropriate alternative instrument can be found that satisfies the completeness conditions. Loosely speaking, such an instrument must be correlated with the bids but independent of the bids conditional on Y. For example, consider a timber auction in which Y denotes the quality of the timber for sale. An instrument in this context might be average rainfall or soil quality, which is related to timber quality but does not directly affect bidders' valuations.

Identification: assumptions and argument When there is unobserved heterogeneity, the unconditional distribution of bids observed by the econometrician differs from the conditional distribution that bidders use when formulating their bids. Thus, correct inference about bidder values requires identification of the joint distribution of bids and the unobserved heterogeneity. We achieve such identification by applying a result from Hu and Schennach (2008). Recall that $g(.|y)$ denotes the conditional pdf of bidder i's bid. For our main identification result in the UH model, we impose the following assumptions:

UH Monotonicity Condition: There exists a known functional M such that $M[g(.|y)]$ is strictly increasing in y.

UH Completeness Condition #1: For all i, k, and $h : \mathcal{B} \to \mathbb{R}$ having bounded conditional expectation, $E\left[h\left(B_k\right)|b_i\right] = 0$ for all $b_i \in [0, \bar{b}(\underline{y})]$ implies that $h\left(b_k\right) = 0$ for all $b_k \in [0, \bar{b}(\underline{y})]$.

UH Completeness Condition #2: For all k and $h : \mathcal{B} \to \mathbb{R}$ having bounded conditional expectation, $E\left[h\left(Y\right)|b_k\right] = 0$ for all $b_k \in [0, \bar{b}(\underline{y})]$ implies that $h\left(y\right) = 0$ for all $y \in \mathcal{Y}$.

[6] Without uniqueness, the equilibrium selected by bidders would be a source of unobserved heterogeneity. Further, the approach developed here *cannot* accommodate this sort of unobserved heterogeneity, since equilibrium bids across multiple equilibria need not be ordered so as to satisfy our monotonicity condition.

Theorem 1 (UH Identification) *Suppose that* $n \geq 3$*. If the UH Monotonicity and Completeness Conditions hold, then the joint distribution of* $(V_1, ..., V_n, Y)$ *is identified from the joint distribution of bids* $(B_1, ..., B_n)$*.*

Proof. The proof has two steps. First, under the assumptions of Theorem 1 (and the maintained assumption of bounded densities, corresponding to Assumption 1 of Hu and Schennach (2008)), the joint distribution of $(B_1, ..., B_n, Y)$ is identified from that of $(B_1, ..., B_n)$. For this, we apply Theorem 1 of Hu and Schennach (2008) ("HS"), using *three bids* (B_i, B_j, B_k) as multiple, conditionally independent measurements of Y. Then, by established methods in the literature (e.g. Guerre, Perrigne, and Vuong (2000)), the joint distribution of values $(V_1, ..., V_n)$ conditional on Y is identified from the distribution of bids $(B_1, ..., B_n)$ conditional on Y, by the first-order conditions of equilibrium bidding in the IPV model. ■

The key step of our identification argument applies Theorem 1 of HS to identify $(B_1, ..., B_n, Y)$ from $(B_1, ..., B_n)$. To shed light on why we require *three* bids and impose the UH Monotonicity and UH Completeness Conditions, we provide a self-contained proof of this key step in the Appendix.

Lemma 1 (Corollary to HS, Theorem 1) *Suppose that* $n \geq 3$*. If the UH Monotonicity and UH Completeness Conditions hold, then* $(B_1, ..., B_n, Y)$ *is identified from the joint distribution of bids* $(B_1, ..., B_n)$*.*

Proof. See the Appendix. ■

2.1 Sufficient conditions for UH Monotonicity

UH Monotonicity is a non-primitive condition on the distribution of equilibrium bids. However, this monotonicity property of *bids* follows directly from our maintained Monotonicity Assumption on *values*. (Proposition 2 is an immediate corollary of Proposition 1.)

Proposition 2 (UH Monotonicity) *When the Monotonicity Assumption is satisfied, the UH Monotonicity Condition is satisfied with respect to the operator corresponding to the maximum of the equilibrium bid support,* $M[g(.|Y = y)] = \max \text{supp}(B_i|Y = y)$*.*

2.2 Sufficient conditions for UH Completeness

The UH Completeness Conditions are also non-primitive conditions on the distribution of equilibrium bids.

To shed light on the economic content of the completeness conditions necessary for identification, we begin by considering the special case of our model in which there is a finite state-space (Section 2.2.1). In this case, completeness follows directly from our Monotonicity Assumption on the underlying distribution of bidder values, so that *no further assumptions are required*. The case in which the state-space is continuous is more technically challenging (Section 2.2.2). For this case, we provide and discuss new sufficient conditions for completeness based on recent findings by Hu and Shiu (2011).

2.2.1 *Finite state-space.*

Suppose that the unobserved heterogeneity takes finitely many values which, without loss of generality, we label as $Y \in \{1, 2, 3, \ldots, K\}$. For convenience, we will also assume that the number of points of support K is known. In this case, our identification result holds under weaker completeness assumptions corresponding to those of Hu (2008). First, a definition is needed.

Definition 1 (Discretization) *Suppose that $\mathcal{Y} = \{1, ..., K\}$. A "discretization" is any monotone, onto mapping $D : \mathcal{B} \to \mathcal{Y}$. Each such mapping is equivalent to a partition of the bid-space $\mathcal{B}$ into K intervals. Let $d_k = D(b_k)$ be shorthand for the interval to which bidder k's bid belongs.*

UH Completeness Condition #1': There exists a discretization such that, for all i, k and all bounded functions $h : \mathcal{Y} \to \mathbb{R}$, $E[h(D_k)|d_i] = 0$ for all $d_i \in \mathcal{Y}$ implies that $h(d_k) = 0$ for all $d_k \in \mathcal{Y}$.

UH Completeness Condition #2': There exists a discretization such that, for all k and all bounded functions $h : \mathcal{Y} \to \mathbb{R}$, $E[h(Y)|d_k] = 0$ for all $d_k \in \mathcal{Y}$ implies that $h(y) = 0$ for all $y \in \mathcal{Y}$.

UH Completeness Conditions #1'-#2' are weaker than Conditions #1-#2. Indeed, these conditions are satisfied whenever the joint distribution of two bidders' discretized bids satisfies a full-rank condition.

Proposition 3 (Completeness: Finite Case) *Suppose that* $Y = \{1, ..., K\}$ *and that there exists a discretization such that*

$$Rank(M_{d_i,d_k}) = K, \tag{1}$$

where $M_{d_i,d_k} = [\Pr\{d_i = i', d_k = k',\}]_{i' \in \mathcal{Y}, k' \in \mathcal{Y}}$. *Then UH Completeness Conditions #1' and #2' are satisfied.*

Proof. The rank condition $Rank(M_{d_i,d_k}) = K$ immediately implies UH Completeness Condition #1'. Further, since $M_{d_i,d_k} = M_{d_i|Y} \times M^T_{d_k,Y}$ and the $K \times K$ matrix M_{d_i,d_k} has rank K by Condition #1', $Rank(M_{d_i,d_k}) = Rank(M_{d_i|Y}) = Rank(M_{d_k,Y}) = K$. Thus, Condition #2' is satisfied. ■

Theorem 2 (UH Identification when $\mathcal{Y}$ is finite) *Suppose that* $n \geq 3$, $\mathcal{Y} = \{1, ..., K\}$, *and the Monotonicity Assumption is satisfied. The joint distribution of values and the state* $(V_1, ..., V_n, Y)$ *is identified from the joint distribution of bids* $(B_1, ..., B_n)$.

Proof. We begin by arguing that the weaker UH Completeness Condition #1' and #2' are enough for identification. The basic idea is that, since the unobserved heterogeneity takes finitely many values, completeness in a finite-dimensional vector space is enough to identify the model. Note that the joint distribution of bids and discretized bids satisfies

$$\begin{aligned} & g\left(D\left(b_i\right) = i', b_j, D\left(b_k\right) = k'\right) \\ = & \sum_y \Pr\left\{D\left(b_i\right) = i' | Y = y\right\} g\left(b_j, Y = y\right) \Pr\left\{D\left(b_k\right) = k' | Y = y\right\} \end{aligned}$$

or, for a given b_j,

$$M_{d_i,b_j,d_k} = M_{d_i|Y} \times D_{b_j,Y} \times M^T_{d_k|Y},$$

where

$$\begin{aligned} M_{d_i|Y} &= [\Pr\{D(b_i) = k | Y = l\}]_{k \in \mathcal{Y}, l \in \mathcal{Y}} \\ D_{b_j,Y} &= diag\{g(b_j, Y = 1), ..., g(b_j, Y = K)\}. \end{aligned}$$

Given UH Monotonicity and the full-rank condition (1), Theorem 1 of Hu (2008) for the discrete case then implies that the distribution of bids conditional on the unobserved heterogeneity, $g(b_j|Y)$, together with the marginal distribution of the unobserved heterogeneity, $\Pr(Y = y)$ for all $y \in \mathcal{Y}$, is nonparametrically identified.

To complete the proof, it suffices to show that the full-rank condition follows from the Monotonicity Assumption. (By Proposition 2, the Monotonicity Assumption implies UH Monotonciity.) By Proposition 1, the maximum $\bar{b}(y)$ of the equilibrium bid support is increasing in the state. Consider the discretization $D : \mathcal{B} \to \{1, ..., K\}$ defined as follows: $D(b) = 1$ for all $b \in (0, \bar{b}(1))$, ... , $D(b) = K - 1$ for all $b \in (\bar{b}(K-2), \bar{b}(K-1))$, and $D(b) = K$ for all $b > \bar{b}(K-1))$. Conditional on $Y = k$, $\text{supp}(D(B)) = \{1, ..., k\}$. Thus, $M_{d_k,Y}$ is a triangular matrix of rank K. The result then follows since the rank of M_{d_i,d_k} is the same as the rank of $M_{d_k,Y}$. ■

Discussion: The proof of Theorem 2 leverages the fact that the maximum of the equilibrium bid support is increasing in the state, but this is not essential. Suppose that bidder values are unbounded, but that the state-space has exactly two elements. For clarity, label these states as "L" and "H". This special case is useful from an expositional point of view, since the full-rank condition (1) reduces to a simple correlation condition, that there exists some threshold $b^* \in \mathcal{B}$ such that

$$\Pr(B_j > b^* | B_i > b^*) \neq \Pr(B_j > b^* | B_i < b^*). \tag{2}$$

This correlation condition follows directly from the Monotonicity Assumption. To see why, note that there must exist some b^* such that $G(b^*|Y = H) < G(b^*|Y = L)$.[7] The correlation condition (2) is therefore satisfied with respect to the discretization $D : \mathcal{B} \to \{L, H\}$ defined as follows: $D(b) = H$ for all $b > b^*$ and $D(b) = L$ for all $b < b^*$.

2.2.2 *Continuous state-space.*

When the state-space is continuous, intuitive sufficient conditions for completeness are more difficult to find. Proposition 4 provides a set of sufficient conditions based on recent findings in Hu and Shiu (2011). (We discuss these conditions at length below.)

Proposition 4 (Completeness: Continuous Case) *Suppose that $g(b, y)$ is continuously differentiable over its support. UH Completeness Condition #2 is satisfied if there exists bid $b_0 \in [0, \bar{b}(\underline{y})]$, convergent sequence $\{b_m : m = 1, 2, ...\} \to b_0$, and open neighborhood $\mathcal{N}(b_0)$ such that*

[7] When the distribution of bidder values is strictly higher in the sense of first-order stochastic dominance, it is easy to show that the distribution of equilibrium bids is not weakly lower in the sense of first-order stochastic dominance.

i) at the limit point b_0, the characteristic function $\phi_{b_0}(t)$ of $g(b_0,\cdot) \in \mathcal{L}^2(\mathbb{R})$ satisfies $0 < |\phi_{b_0}(t)| < Ce^{-\delta|t|}$ for some constants C and $\delta > 0$;

ii) $\frac{\partial}{\partial b} g(b,\cdot) \in \mathcal{L}^2(\mathbb{R})$ for $b \in \mathcal{N}(b_0)$ and $\frac{\partial}{\partial y} g(b_0,\cdot) \in \mathcal{L}^2(\mathbb{R})$;

iii) the sequence $\{g(b_m, y) : m = 1, 2, ...\}$ is linearly independent, i.e., for all finite I,

$$\sum_{i=1}^{I} c_i g(b_i, y) = 0 \text{ for all } y \in \mathbb{R} \text{ implies } c_i = 0 \text{ for all } i = 1, 2, ..., I.$$

UH Completeness Condition #1 is satisfied if, in addition to conditions i), ii), and iii), there exists y_0, convergent sequence $\{y_m : m = 1, 2, ...\} \to y_0$, and open neighborhood $\mathcal{N}(y_0)$ such that

i') at the limit point y_0, the characteristic function $\phi_{y_0}(t)$ of $g(\cdot, y_0) \in \mathcal{L}^2(\mathbb{R})$ satisfies $0 < |\phi_{y_0}(t)| < Ce^{-\delta|t|}$ for some constants C and $\delta > 0$;

ii') $\frac{\partial}{\partial y} g(\cdot, y) \in \mathcal{L}^2(\mathbb{R})$ for $y \in \mathcal{N}(y_0)$ and $\frac{\partial}{\partial b} g(\cdot, y_0) \in \mathcal{L}^2(\mathbb{R})$;

iii') the sequence $\{g(b, y_m) : m = 1, 2, ...\}$ is linearly independent, i.e., for all finite I,

$$\sum_{i=1}^{I} c_i g(b, y_i) = 0 \text{ for all } b \in \mathcal{B} \text{ implies } c_i = 0 \text{ for all } i = 1, 2, ..., I. \quad (3)$$

For an idea why conditions (i)-(iii) imply UH Completeness Condition #2, note that he completeness of $g(b, y)$ for all b is implied by the completeness of $g(b, y)$ for b in some open neighborhood $N(b_0)$. Given conditions (i)-(iii), it is possible to construct a complete sequence of densities $\{\hat{g}(b_m, y) : m = 1, 2, ...\}$ which is "close" to the sequence $\{g(b_m, y) : m = 1, 2, ...\}$. This implies the completeness of the density of interest $g(b, y)$ in an open neighborhood $N(b_0)$, which in turn implies the completeness of $g(b, y)$ for all b and hence UH Completeness Condition #2. Similarly, conditions (i')-(iii') imply UH Completeness Condition #1.

Proof. Without loss, suppose that (B_i, Y) has support $\mathbb{R} \times \mathbb{R}$.[8] To establish UH Completeness Condition #2, it suffices to show that the family $\{g(b_m, y) : m = 1, 2, ...\}$ is complete in $\mathcal{L}^2(\mathcal{Y})$. (The family $\{g(b_m, y) : m = 1, 2, ...\}$ is complete if and only if $\{g(y|b_m) : m = 1, 2, ...\}$ is complete.)

[8] The UH Completeness Conditions are imposed on the common part of the equilibrium bid support, $[0, \overline{b}(\underline{y})]$. One can smoothly transform this interval into the real line.

Theorem 2.1 in Hu and Shiu (2011) establishes that conditions i), ii), and iii) imply completeness of the family $\{g(b_m, y) : m = 1, 2, ...\}$. We briefly describe the proof here and refer to that paper for the details. Since $\{g(b_m, y) : m = 1, 2, ...\}$ is a linearly independent sequence in a normed space, it contains a so-called $\omega-$independent subsequence $\{g(b_{m'}, y) : m' = 1, 2, ...\}$. Condition i) implies that there exists a complete sequence of functions $\{\hat{g}(b_{m'}, y) : m' = 1, 2, ...\}$ such that $\hat{g}(b, y) = g(b_0, y - \mu(b))$ with $\mu(b_0) = 0$. Condition ii) implies that the total deviation from the complete sequence $\{\hat{g}(b_{m'}, y) : m' = 1, 2, ...\}$ to the original sequence $\{g(b_{m'}, y) : m' = 1, 2, ...\}$ is finite, in the sense that

$$\sum_{m'=1}^{\infty} \frac{\|g(b_{m'}, \cdot) - \hat{g}(b_{m'}, \cdot)\|}{\|\hat{g}(b_{m'}, \cdot)\|} < \infty. \tag{4}$$

The key step of the proof is to show that, for a complete sequence $\{\hat{g}(b_{m'}, y) : m' = 1, 2, ...\}$ and the $\omega-$independent sequence $\{g(b_{m'}, y) : m' = 1, 2, ...\}$, equation (4) implies that the sequence $\{g(b_{m'}, y) : m' = 1, 2, ...\}$ is complete and, therefore, that $\{g(b, y) : b \in \mathcal{N}(b_0)\}$ is complete. (We omit details for this step.)

Next, consider UH Completeness Condition #1. It suffices to establish completeness of $g(b_k, b_i)$, where

$$g(b_k, b_i) = \int g(b_k|Y)g(Y, b_i)dY. \tag{5}$$

We have already argued that conditions i), ii), and iii) are sufficient for completeness of $\{g(b_m, y) : m = 1, 2, ...\}$. To complete the proof, we need only to show that i'), ii'), and iii') are sufficient conditions for the completeness of $\{g(b, y_m) : m = 1, 2, 3, ...\}$. Yet this follows from a nearly identical argument as above, the only difference being that we consider the sequence $\{g(b, y_m) : m = 1, 2, ...\}$ in place of the sequence $\{g(b_m, y) : m = 1, 2, ...\}$. ■

Discussion of Proposition 4. Intuitively, we may consider the joint density $g(b_0, y)$ as the limit of the sequence of functions $\{g(b_m, y) : m = 1, 2, ...\}$ *or* as the limit of the sequence $\{g(b_0, y - \mu(b_m)) : m = 1, 2, ...\}$, where $\mu(b_0) = 0$. Thus, we may think of the sequence $\{g(b_m, y) : m = 1, 2, ...\}$ as being "close" to taking the form of a convolution in a neighborhood of the limit point b_0. Completeness for this nearby convolution distribution follows as usual, from a non-vanishing characteristic function assumption. The key insight of Hu and Shiu (2011) is that, under the additional assumptions of Proposition 4, such "local closeness" to a complete distribution is itself sufficient for completeness.

While we hope that the preceding intuition is helpful, we would like to emphasize that the joint density $g(b, y)$ does not need to be anything like a convolution. Completeness holds whenever conditions i)-iii) hold *locally* for one limit point b_0 and conditions i')-iii') hold *locally* for one limit point y_0. That said, the assumptions of Proposition 4 do impose substantive restrictions on the joint distribution of bids and the state, as we shall now discuss.

Conditions i) and i'). Condition i') requires that there exists some state-level y_0 such that the distribution of a bid conditional on y_0 has a non-vanishing and exponentially decaying characteristic function. The assumption that the ch.f. has no zeros on the real line is standard in the deconvolution literature, and is also a necessary condition in (say) Li (2002) and Krasnokutskaya (2011). The assumption that the ch.f. has exponential decay is only restrictive on the tails, i.e., as $|t| \to \infty$. Roughly speaking, this assumption rules out distributions with thin tails, such as uniform distributions and exponential distributions, but allows normal distributions and Cauchy distributions. Exponential decay is needed to obtain completeness on $\mathcal{L}^2(\mathbb{R})$. However, the exponential decay assumption can be relaxed if one only considers completeness on the subset of functions in $\mathcal{L}^2(\mathbb{R})$ having exponentially decaying Fourier transforms. Condition i) is similar, except that it applies to the distribution of the state conditional on bid b_0.

Conditions ii) and ii'). These assumptions are innocuous. Indeed, as long as the joint density $g(v_i, y)$ is continuously differentiable in y (as well as in v_i, as we have assumed), Lebrun (2002) shows that equilibrium bidding strategies are continuously differentiable in y as well as in v_i. In this case, condition ii) holds for *all* $b_0 \in \mathcal{B}$ while condition ii') holds for *all* $y_0 \in \mathcal{Y}$.

Conditions iii) and iii'). These linear independence assumptions also appear to be mild. Note that conditions iii) or iii') are only violated if there exist a *finite* number of constants c_i satisfying the *infinite* system of equations (3). (See Hu and Shiu (2011) for a variety of sufficient conditions for linear independence.)

3 Extension: conditionally independent private values

Our results also apply to a setting in which bidders have conditionally independent private values ("CIPV model"). The main behavioral distinction between the UH and CIPV models is that, in the UH model, bidders observe Y before they choose their bid while, in the CIPV model, bidders do not observe Y. As we shall see below, this distinction makes it *easier* to analyze the CIPV model, as well as to provide sufficient conditions for completeness.

As before, we achieve identification by applying a result from Hu and Schennach (2008). The main difference is that our monotonicity and completeness assumptions are now *primitive* conditions on bidder values, rather than non-primitive conditions on equilibrium bids.

CIPV Monotonicity Condition: There exists a known functional M such that $M[f(.|y)]$ is strictly increasing in y.

CIPV Completeness Condition #1: For all i, k and all $h : \mathbb{R} \rightarrow \mathbb{R}$ having bounded conditional expectation, $E[h(V_k)|v_i] = 0$ for all v_i implies that $h(v_k) = 0$ for all v_k.

CIPV Completeness Condition #2: For all k and all $h : \mathbb{R} \rightarrow \mathbb{R}$ having bounded conditional expectation, $E[h(Y)|v_k] = 0$ for all v_k implies that $h(y) = 0$ for all $y \in \mathcal{Y}$.

Theorem 3 (CIPV Theorem) *Suppose that $n \geq 3$. If the CIPV Monotonicity and CIPV Completeness Conditions hold, then the joint distribution of $(V_1, ..., V_n, Y)$ is identified from the joint distribution of bids $(B_1, ..., B_n)$.*

Proof. The proof has two steps. First, by established methods in the literature (e.g. Li, Perrigne, and Vuong (2000)), the joint distribution of values $(V_1, ..., V_n)$ is identified from the distribution of bids $(B_1, ..., B_n)$, by the first-order conditions of equilibrium bidding in the CIPV model. The second step is to show that, under the assumptions of the CIPV Theorem, the joint distribution of values and the state $(V_1, ..., V_n, Y)$ is identified from $(V_1, ..., V_n)$. For this step, we apply Theorem 1 of HS, where three values (V_i, V_j, V_k) serve as multiple, conditionally independent measurements of Y. ■

Sufficient condition for CIPV Monotonicity. As in the UH model, it suffices for the distribution of bidder values to be FOSD-increasing in Y.

Proposition 5 (CIPV Monotonicity) *Suppose that the Monotonicity Assumption is satisfied. The CIPV Monotonicity Condition is then satisfied with respect to the mean operator, $M[g(.|Y = y)] = E[B_i|Y = y]$.*

Proof. In the CIPV model, bidders' bids depend on their realized values but not the state. Thus, the distribution of equilibrium bids is just a monotone transformation of the distribution of bidder values, where this transformation does not vary with the state. Since the distribution of bidder values is FOSD-increasing in the state by the Monotonicity

Assumption, the distribution of equilibrium bids must also be FOSD-increasing in the state. In particular, the mean equilibrium bid is strictly increasing in the state. ■

Sufficient conditions for CIPV Completeness. The CIPV Completeness Conditions are identical to the UH Completeness Conditions in all but one respect: the CIPV Conditions apply to the *primitive* joint distribution of the state and bidder values while the UH Conditions apply to the non-primitive joint distribution of the state and equilibrium bids. Thus, our results on completeness in the UH model translate directly to the CIPV model, once interpreted as applying to values rather than bids. First, when $\mathcal{Y}$ has finite support, CIPV Completeness holds whenever the joint distribution of two bidders' (discretized) values satisfies a full-rank condition analogous to that of equation (1). Second, when $Y \subset \mathbb{R}$, CIPV Completeness is satisfied whenever conditions analogous to conditions i)-iii) and i')-iii') of Proposition 4 are satisfied.

4 Additional extensions

This section aims to illustrate the breadth of application of our identification approach, by examining two settings *outside of the model of Section 2* with independent private values and unobserved heterogeneity. In Section 4.1, we consider a situation in which the seller's (implicit) reserve price is known to the bidders but not observed by the econometrician. The reserve price affects the distribution of equilibrium bids and therefore constitutes a source of unobserved heterogeneity, even if it is uncorrelated with the distribution of bidder values. Given the joint distribution of at least three bids, we identify the distribution of the reserve and the distribution of bidder values. In Section 4.2, we consider a situation in which bidding is costly, but the econometrician does not observe the cost of bidding. Given the joint distribution of at least three bids, we identify the distribution of the cost of bidding and the distribution of bidder values.

4.1 Implicit reserve price as unobserved heterogeneity

Auction data-sets often include the *explicit* reserve price, the minimal bid permitted by the auction rules. However, real-world sellers sometimes refuse to sell to the highest bidder, unless the highest bid exceeds an even higher *implicit* reserve price.[9] This observation has motivated an important empirical literature on auctions with a "random reserve price"; see e.g. Li and Perrigne (2003). Papers in this literature presume that bidders do not know the seller's implicit reserve price when bidding, only the distribution from which it is drawn. Yet sellers can have an incentive to reveal their implicit reserve price to bidders prior to the bidding.[10] If bidders observe the implicit reserve price, the distribution of equilibrium bids will vary with the implicit reserve, making this a potentially important source of unobserved heterogeneity.

We consider here the simplest case in which (i) the realized implicit reserve price $R = r$ is common knowledge among the bidders prior to the bidding, (ii) bidder values are independent of R, and (iii) the econometrician knows nothing about the implicit reserve, not even the distribution from which it is drawn. Such econometrician ignorance could arise naturally, if the seller's implicit reserve depends on his own cost but the econometrician has no data on seller cost.

For simplicity of the exposition, we shall henceforth suppress the distinction between explicit and implicit reserve prices, treating the reserve price as if it is an explicit reserve that is unobserved by the econometrician.[11]

Monotonicity. Equilibrium bids vary monotonically with the reserve price, so that the UH Monotonicity Condition is satisfied. In particular, it is easy to show that the minimum and the maximum of the equilibrium bid support are each strictly increasing in the realized reserve price. Thus, the UH Monotonicity Condition is satisfied with respect to either of the functionals corresponding to the minimum or the maximum of the equilibrium bid support.

[9] If the seller is bound to accept the highest bid whenever it exceeds the explicit reserve, the implicit reserve equals the explicit reserve and there is no problem. Otherwise, the "true" reserve typically exceeds the explicit reserve, and bidders' beliefs about the reserve price affect the distribution of equilibrium bids.

[10] For example, suppose that the seller's cost is random and unobserved by the bidders. Since the optimal reserve price varies with the seller's cost, the seller cannot implement the optimal auction unless he credibly reveals the reserve price to bidders. Consequently, any seller who can commit to an optimal implicit reserve price will always choose to reveal it to bidders. See also Brisset and Naegelen (2006) for another context in which the seller chooses to reveal the reserve price.

[11] When R is an implicit reserve price, the data will include *permissible but unacceptable* bids between the explicit and implicit reserves. R is then the minimal acceptable bid, rather than the minimal permissible bid. The minimal acceptable bid can be inferred from data on bids and auction outcomes, as long as one observes when the seller refuses to sell to the highest bidder.

Completeness. Suppose first that the reserve has finite support, $\text{supp}(R) = \{r_1, ..., r_K\}$. Since the maximum of the equilibrium bid support is increasing in the reserve, UH Completeness Conditions #1'-#2' are each automatically satisfied by the argument of Proposition 3 and Theorem 2.

Suppose next that the reserve has continuous bounded support, $\text{supp}(R) = [\underline{r}, \overline{r}]$, where $\overline{r} < \overline{v}$ so that the object is always sold with positive probability. Unlike in the quality application of Section 2, the conditional distribution of equilibrium bids need not have any shared support over which to impose UH Completeness Conditions #1-#2. To apply our methodology, an additional restriction is needed in order to guarantee that there is a shared support. In particular, since the equilibrium bid support takes the form $[r, \overline{b}(r)]$ for all reserve prices r, it suffices to assume that $\overline{b}(\underline{r}) > \overline{r}$.[12] UH Completeness Conditions #1-#2, imposed now over the shared support $[\overline{r}, \overline{b}(\underline{r})]$, will be satisfied under a set of conditions analogous to those of Proposition 4.

Identification. For ease of the exposition, consider the simpler case in which the reserve price has finite support, $\text{supp}(R) = \{r_1, ..., r_K\}$. As noted above, the UH Monotonicity Condition and the UH Completeness Conditions #1'-#2' are automatically satisfied. Thus, we may apply Theorem 2 to identify the joint distribution of the bids and the unobserved state. More precisely, we may identify the joint distribution of $(B_1, ..., B_n, \gamma(R))$, where $\gamma : \{r_1, ..., r_K\} \to \{1, ..., K\}$ is a normalization.

Identifying the *unnormalized* support of the unobserved heterogeneity requires extra work which, in this case, is trivial and immediate. Namely, since the realized reserve price r_k is the minimum of the support of submitted bids conditional on $\gamma(R) = k$, we may infer r_k directly from the distribution of bids conditional on $\gamma(R) = k$. Thus, in fact, the joint distribution of $(B_1, ..., B_n, R)$ is identified.

Identifying bidder values is now straightforward. In particular, the conditional distribution of $(V_1, ..., V_n)|R$ is identified as usual from that of $(B_1, ..., B_n)|R$, by the first-order conditions of equilibrium bidding as in Guerre, Perrigne, and Vuong (2000). (More precisely, for each realized reserve price $R = r_k$, we may identify the distribution of values above r_k.)

[12] $\overline{b}(\underline{r}) = \overline{v} - \int_{\underline{r}}^{\overline{v}} F(v|y)^{n-1} dv$. Thus, this additional assumption can be viewed as a joint restriction on the reserve-price range $[\underline{r}, \overline{r}]$ and on the distribution of bidder values.

4.2 Bidding cost as unobserved heterogeneity

Samuelson (1985) noted that "competing firms must bear significant bid-preparation and documentation costs" in order to bid in an auction, spawning a large literature on auctions with costly bidding. Suppose that, as in Samuelson (1985), each bidder costlessly learns his private value and then simultaneously decides whether to pay $C \geq 0$ to submit a bid in a first-price auction with zero reserve price. The distribution of equilibrium bids varies with the cost of bidding, making this a potentially important source of unobserved heterogeneity if the cost of bidding varies across auctions but is not observed by the econometrician. (Such a case of unobserved heterogeneity is also discussed in Li and Zheng (2009).)

Monotonicity. Equilibrium bids vary monotonically with the cost of bidding, so that the UH Monotonicity Condition is satisfied. In particular, it is easy to show that the probability of bidding and the maximum of the equilibrium bid support are each strictly decreasing in the cost of bidding C. Thus, the UH Monotonicity Condition is satisfied with respect to either of the functionals corresponding to the probability of bidding or the maximum of the equilibrium bid support, when the space of possible costs is endowed with the reverse order.

Completeness. Suppose first that the cost of bidding is drawn from finite support, $\text{supp}(C) = \{c_1, ..., c_K\}$. Since the maximum of the equilibrium bid support is monotone in the cost of bidding, UH Completeness Conditions #1'-#2' are each automatically satisfied by the argument of Proposition 3 and Theorem 2.

Suppose next that the cost of bidding has continuous bounded support, $\text{supp}(C) = [\underline{c}, \overline{c}]$. For each realized cost c, the equilibrium bid support takes the form $[0, \overline{b}(c)]$, where $\overline{b}(c)$ is decreasing in c. So, $[0, \overline{b}(\overline{c})]$ is part of the equilibrium bid support for all costs $c \in \text{supp}(C)$. All discussion of the completeness conditions in Section 2.2 therefore applies here as well, once the UH Completeness Conditions #1-#2 are imposed instead over the shared support $[0, \overline{b}(\overline{c})]$. In particular, these conditions will be satisfied under a set of conditions analogous to those of Proposition 4.

Identification. For ease of the exposition, consider the simpler case in which the cost of bidding has finite support, $\text{supp}(C) = \{c_1, ..., c_K\}$. As noted above, the UH Monotonicity Condition and the UH Completeness Conditions #1'-#2' are automatically satisfied. Thus, we may apply Theorem 2 to identify the joint distribution of the bids and the unobserved state. That is, we may identify the joint distribution of $(B_1, ..., B_n, \gamma(C))$, where $\gamma : \{c_1, ..., c_K\} \to \{1, ..., K\}$ is a normalization.

Identifying the *unnormalized* support of the unobserved heterogeneity requires extra work which, in this case, is not as immediate as in the reserve price example of Section 4.1. First, for every realization of the normalized cost of bidding $k = 1, ..., K$, the conditional distribution of values $(V_1, ..., V_n)|(\gamma(C) = k)$ is identified from the conditional distribution of bids $(B_1, ..., B_n)|(\gamma(C) = k)$, by the first-order conditions of equilibrium bidding as in Guerre, Perrigne, and Vuong (2000). More precisely, the distribution of values is identified above the minimal value $\underline{v}(c_k)$ given which each bidder submits a bid, conditional on bidding cost $C = c_k$.

The threshold $\underline{v}(c_k)$ is determined by the indifference condition that

$$c_k = \underline{v}(c_k) F(\underline{v}(c_k))^{n-1} \text{ for all } k = 1, ..., K. \tag{6}$$

(A bidder having value $v_i = \underline{v}(c_k)$ bids zero in equilibrium, wins with probability $F(\underline{v}(c_k))^{n-1}$, and is indifferent between bidding or not.) Both the probability of non-bidding $F(\underline{v}(c_k))$ and the bidding threshold $\underline{v}(c_k)$, conditional on $C = c_k$, are identified from the distribution of bidder values conditional on $\gamma(C) = k$. Equation (6) therefore allows us to identify c_k from the distribution of bidder values conditional on $\gamma(C) = k$. Thus, in fact, the joint distribution of $(V_1, ..., V_n, C)$ is identified.

5 Concluding Remarks

This paper has developed a novel approach to identify first-price auction models with independent private values in the face of one-dimensional unobserved heterogeneity. Our key identifying assumption is that the distribution of *bidder values* is increasing in the state, in the sense of first-order stochastic dominance. When the state-space is finite, this monotonicity assumption suffices to imply both the monotonicity and the completeness conditions on the distribution of *equilibrium bids* that are necessary for our identification approach. When the state-space is continuous, additional assumptions are required for completeness but we provide new sufficient conditions based on recent findings by Hu and Shiu (2011).

Our identification approach can be adapted to a wide variety of auction environments, in which some location of the distribution of equilibrium bids is increasing in the unobserved state. We consider three such applications, when (i) bidders also do not observe the underlying state, so that the model is one of conditionally independent private values (Section 3); (ii) the seller's implicit reserve price is known to the bidders but unobserved by the econometrician (Section 4.1); and (iii) the bidders' cost of preparing a bid is known to the bidders but unobserved by the econometrician (Section 4.2).

We see three important directions for future work building upon this paper. First, in many applications it is likely that there are multiple potential sources of unobserved heterogeneity. We are currently exploring an extension of this paper's analysis to a setting with multi-dimensional unobserved heterogeneity. Second, the proof of our main result relies heavily on the assumption of *independent* private values. We are working to extend our results to settings in which bidders have affiliated private values conditional on the state. Finally, we are considering how to extend our results to "endogenous participation" models, roughly defined as models of entry in auctions. Unlike the "bidding cost" example of Section 4.2, bidders' entry decisions in such models depend on unobserved auction characteristics which also affect their valuations; see e.g. Haile, Hong, and Shum (2003) and Li and Zheng (2009).

Appendix: Proof of Lemma 1

Proof. Since bidders are symmetric and equilibrium bids are conditionally independent, the joint distribution of $(B_1, ..., B_n, Y)$ is identified from the distributions of Y and $B_k|Y$, for any $k = 1, ..., n$. Fix any three bidders i, j, k. (The following proof can be repeated for any triplet of bidders.) We will show that the distributions of Y and $B_k|Y$ are identified from the joint distribution of (B_i, B_j, B_k).

The following integral operators will be useful in the proof:

$$\left(L_{B_k|Y} h\right)(b_k) = \int g_{B_k|Y}(b_k|y)h(y)dy$$

$$\left(L_{B_i|B_k} h\right)(b_i) = \int g_{B_i|B_k}(b_i|b_k)h(b_k)db_k$$

$$\left(L_{B_k,\bar{b}_j|B_i} h\right)(b_k) = \int g_{B_k,B_j|B_i}(b_k, \bar{b}_j|b_i)h(b_i)db_i$$

$$\left(D_{\bar{b}_j|Y} h\right)(y) = g_{B_j|Y}(\bar{b}_j|y)h(y)$$

$$\left(L_{Y|B_i} h\right)(y) = \int g_{Y|B_i}(y|b_i)h(b_i)db_i.$$

In the above and in what follows, variables with bars ($^-$) denote fixed values.

Injectivity of $L_{B_k|Y}$ and $L_{B_i|B_k}$.

Note that

$$\begin{aligned}
E\left[h\left(Y\right)|b_k\right] &= \int g_{Y|B_k}(y|b_k)h(y)dy \\
&= \frac{1}{g_{B_k}(b_k)} \int g_{B_k|Y}(b_k|y)\left[g_Y(y)h(y)\right]dy \\
&\equiv \frac{1}{g_{B_k}(b_k)} \int g_{B_k|Y}(b_k|y)\hat{h}(y)dy \\
&= \frac{1}{g_{B_k}(b_k)} \left(L_{B_k|Y}\hat{h}\right)(b_k)
\end{aligned}$$

where we have set $\hat{h}(y) = g_Y(y)h(y)$. By UH Completeness #2, $E\left[h\left(Y\right)|b_k\right] = 0$ for all b_k on the shared support $[0, \bar{b}(\underline{y})]$ implies that $h\left(y\right) = 0$ for all y. Thus, the linear operator $L_{B_k|Y}$ is injective. Similarly, UH Completeness #1 implies that $L_{B_i|B_k}$ is injective (details omitted).

Eigenvalue/eigenfunction decomposition.

Since bidder values are assumed to be conditionally independent, bids are also independent conditional on Y:

$$g_{B_k,B_j|B_i}(b_k, b_j|b_i) = \int g_{B_k|Y}(b_k|y)g_{B_j|Y}(b_j|y)g_{Y|B_i}(y|b_i)dy. \tag{7}$$

In particular, we may express the operator $L_{B_k,\bar{b}_j|B_i}$ as follows:

$$\begin{aligned}
\left(L_{B_k,\bar{b}_j|B_i}h\right)(b_k) &= \int g_{B_k,B_j|B_i}(b_k, \bar{b}_j|b_i)h(b_i)db_i \\
&= \int \left(\int g_{B_k|Y}(b_k|y)g_{B_j|Y}(\bar{b}_j|y)g_{Y|B_i}(y|b_i)dy\right)h(b_i)db_i \\
&= \int g_{B_k|Y}(b_k|y)g_{B_j|Y}(\bar{b}_j|y)\left(\int g_{Y|B_i}(y|b_i)h(b_i)db_i\right)dy \\
&= \int g_{B_k|Y}(b_k|y)g_{B_j|Y}(\bar{b}_j|y)\left[\left(L_{Y|B_i}h\right)(y)\right]dy \\
&= \int g_{B_k|Y}(b_k|y)\left[\left(D_{\bar{b}_j|Y}L_{Y|B_i}h\right)(y)\right]dy \\
&= \left(L_{B_k|Y}D_{\bar{b}_j|Y}L_{Y|B_i}h\right)(b_k).
\end{aligned}$$

In other words,

$$L_{B_k,\bar{b}_j|B_i} = L_{B_k|Y}D_{\bar{b}_j|Y}L_{Y|B_i}. \tag{8}$$

Next, integrating out b_j in (7), $g_{B_k|B_i}(b_k|b_i) = \int g_{B_k|Y}(b_k|y)g_{Y|B_i}(y|b_i)dy$. Mimicking the steps above, we may now similarly show that

$$L_{B_k|B_i} = L_{B_k|Y}L_{Y|B_i}. \tag{9}$$

Since $L_{B_k|Y}$ is injective,

$$L_{Y|B_i} = L_{B_k|Y}^{-1} L_{B_k|B_i}. \tag{10}$$

Substituting this expression in (8), for any fixed $\bar{b}_j$, yields

$$L_{B_k,\bar{b}_j|B_i} = L_{B_k|Y} D_{\bar{b}_j|Y} L_{B_k|Y}^{-1} L_{B_k|B_i}. \tag{11}$$

By the injectivity of $L_{B_k|B_i}$, finally,

$$L_{B_k,\bar{b}_j|B_i} L_{B_k|B_i}^{-1} = L_{B_k|Y} D_{\bar{b}_j|Y} L_{B_k|Y}^{-1}. \tag{12}$$

This equation implies that the observed LHS has an eigenvalue and eigenfunction decomposition. The eigenvalues are $g_{B_j|Y}(\bar{b}_j|y)$ in the diagonal operator $D_{\bar{b}_j|Y}$ and the eigenfunctions are $g_{B_k|Y}(\cdot|y)$ in the operator $L_{B_k|Y}$. The realization of unobserved heterogeneity y is the index for the eigenvalues and eigenfunctions. UH Monotonicity implies that the eigenvalues are distinctive.

The conditional density $g_{B_k|Y}(\cdot|y)$ of bidder k's bid is identified from this eigenfunction decomposition. Also, both $g_{Y|B_i}(y|b_i)$ and $g_Y(y) = \int g_{Y|B_i}(y|b_i) g_{B_i}(b_i) db_i$ are identified from (10). This completes the proof. ■

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