Econometric Theory, **34**, 2018, 166–185.
doi:10.1017/S0266466617000081

CLOSED-FORM IDENTIFICATION OF DYNAMIC DISCRETE CHOICE MODELS WITH PROXIES FOR UNOBSERVED STATE VARIABLES

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Proxies for unobserved skills and technologies are increasingly available in empirical data. For dynamic discrete choice models of forward-looking agents where a continuous state variable is unobserved but its proxy is available, we derive closed-form identification of the structure by explicitly solving integral equations. In the first step, we derive closed-form identification of Markov components, including the conditional choice probabilities and the law of state transition. In the second step, we plug in these first-step identifying formulas to obtain primitive structural parameters of dynamically optimizing agents.

1. INTRODUCTION

The structure of forward-looking agents making dynamic decisions based on unobserved state variables is of wide interest in economic research. Furthermore, many recent economic studies are concerned with the dynamics of unobserved state variables, such as human capital stocks or technologies (e.g., Cunha, Heckman, and Schennach, 2010; Todd and Wolpin, 2012). While econometricians may not observe the true state variables, they often have access to proxy variables for these latent variables. Cunha et al. (2010), for example, obtain a list of proxies for cognitive skills, such as the Peabody Individual Achievement Test (PIAT) scores available in NLSY79, and a list of proxies for noncognitive skills, such as the Behavior Problems Index (BPI) available in NSLY79, for their analysis of dynamic skill production that takes parental investment as input. With the increasing availability of proxies in empirical data, it is a natural idea to use them to identify and estimate primitive structural parameters for dynamically optimizing agents, such as parents investing in their children. Because of the nonlinearity

The authors can be reached at yhu@jhu.edu and sasaki@jhu.edu. We benefited from useful comments by the editor (Peter C.B. Phillips), the coeditor (Arthur Lewbel), three anonymous referees, seminar participants at Cambridge, Centre de Recherche en Économie et Statistique Paris, George Washington, London School of Economics, Oxford, Rice, Shanghai University of Finance and Economics, Texas A&M, Toulouse School of Economics, University College London, Wisconsin-Madison, 2013 Greater New York Metropolitan Area Econometrics Colloquium, and 2014 Shanghai Econometrics Workshop. The usual disclaimer applies. Address correspondence to Yingyao Hu, Department of Economics, Wyman Park Building 544E, 3100 Wyman Park Drive, Baltimore, MD 21211; e-mail: yhu@jhu.edu.

of the forward-looking discrete choice structure, however, naive substitution of a proxy generally biases the estimates of structural parameters, even if the proxy entails only a classical measurement error. As such, a proper identification result needs to be established. In this paper, we develop closed-form identification of dynamic discrete choice models when a proxy for an unobserved continuous state variable is available.

The setup for the econometric model and our results are as follows. Agent j at time t makes discrete decisions $d_{j,t}$ based on its latent state $x^*_{j,t}$, accounting for the future evolution of $x^*_{j,t}$ which may be affected by $d_{j,t}$. The state $x^*_{j,t}$ is observed by agent j, but not by econometricians. We, as econometricians, obtain a proxy $x_{j,t} = x^*_{j,t} + \varepsilon_{j,t}$ for the unobserved state $x^*_{j,t}$ with a ($d_{j,t}$-conditionally) independent measurement error $\varepsilon_{j,t}$. If true state $x^*_{j,t}$ were observable, then identification of the structural parameters of forward-looking agents would follow from identification of two auxiliary objects: (1) the conditional choice probability (CCP) denoted by $f_1(d_t \mid x^*_t)$; and (2) the law of state transition denoted by $f_2(x^*_t \mid d_{t-1}, x^*_{t-1})$ (Hotz and Miller, 1993). Our result claims that these two auxiliary objects, $f_1(d_t \mid x^*_t)$ and $f_2(x^*_t \mid d_{t-1}, x^*_{t-1})$, are identified using proxies $x_{j,t}$ without observing true states $x^*_{j,t}$, and, consequently, the structural parameters of current–time payoff are identified.

Indeed, identification of dynamic discrete choice models with unobservables (e.g., Aguirregabiria and Mira, 2007; Kasahara and Shimotsu, 2009; Arcidiacono and Miller, 2011; Hu and Shum, 2012 – see also the survey by Aguirregabiria and Mira, 2013) and identification of dynamic discrete choice models with continuous state variables (Srisuma and Linton, 2012) are studied in the literature, but no preceding work handles continuous unobserved state variables. The model studied in this paper allows for continuously distributed unobservables at the expense of the requirement of proxy variables for the unobservables. Our use of proxy variables in dynamic structural models is related to Cunha et al. (2010) and Todd and Wolpin (2012). As we are interested in the payoff parameters of forward-looking agents characterized by Bellman equations, however, we follow a different approach as outlined below.

In the first step, we identify the CCP $f_1(d_t \mid x^*_t)$ and the law of state transition $f_2(x^*_t \mid d_{t-1}, x^*_{t-1})$ using a proxy variable in Section 2. For this step, we use an approach related to the closed-estimator of Schennach (2004b) and Hu and Sasaki (2015) for nonparametric regression models with measurement errors (cf. Li, 2002; Schennach, 2004a), as well as the deconvolution methods (Li and Vuong, 1998). In the second step, the preliminarily identified Markov components, $f_1(d_t \mid x^*_t)$ and $f_2(x^*_t \mid d_{t-1}, x^*_{t-1})$, are used in turn to identify structural parameters of a current-time payoff. Once the Markov components are identified, this second step can be conducted by directly applying existing CCP-based methods (Hotz and Miller, 1993; Hotz, Miller, Sanders and Smith, 1994). Section 3 presents a brief explanation of this second-step procedure, following a state-of-the-art technique (Srisuma and Linton, 2012; Srisuma, 2015) in the literature that nicely handles continuous state variables.

2. CLOSED-FORM IDENTIFICATION OF MARKOV COMPONENTS

2.1. The Model and Notations

Our basic notations are fixed as follows. A discrete control variable, taking values in $\{0, 1, \ldots, \bar{d}\}$, is denoted by $d_{j,t}$. For example, it may indicate the number of books targeted at age t that child j has, as a measure of parental investment in child of age t (Cunha et al., 2010). An unobserved state variable is denoted by $x^*_{j,t}$. It may, for example, be the reading skills of child j at age t. Finally, a proxy for $x^*_{j,t}$ is denoted by $x_{j,t}$. It may, for example, be the PIAT score for reading comprehension of child j at age t, as in Cunha, Heckman, and Schennach. We consider the dynamics of this list of random variables. To simplify exposition, we hereafter omit individual subscript j. Our identification strategy is based on the assumptions listed below.

Assumption 1 (First-order Markov process)**.** The triple $\{d_t, x^*_t, x_t\}$ jointly follows a first-order Markov process.

This Markovian structure is decomposed into three independent modules, as follows.

Assumption 2 (Independence)**.** The Markov kernel can be decomposed as

$$f\left(d_t, x^*_t, x_t | d_{t-1}, x^*_{t-1}, x_{t-1}\right) = f_1\left(d_t | x^*_t\right) f_2\left(x^*_t | d_{t-1}, x^*_{t-1}\right) f_3\left(x_t | x^*_t\right),$$

where the three components represent

$f_1\left(d_t \vert x^*_t\right)$	conditional choice probability (CCP);
$f_2\left(x^*_t \vert d_{t-1}, x^*_{t-1}\right)$	transition rule for the unobserved state variable; and
$f_3\left(x_t \vert x^*_t\right)$	proxy model.

This independence assumption is the key to our closed-form identification results. To better understand this key assumption in the context of the standard independence assumptions used in the dynamic discrete choice literature, we discuss primitive conditions for this decomposition. Consider the following four independence conditions.

(i) Rust's conditional independence assumption:

$$f(d_t, x^*_t \mid d_{t-1}, x^*_{t-1}) = f_1(d_t \mid x^*_t) \cdot f_2(x^*_t \mid d_{t-1}, x^*_{t-1}).$$

(ii) True state x^*_t is a sufficient statistic for proxy x_t:

$$f(x_t \mid d_t, x^*_t, d_{t-1}, x^*_{t-1}, x_{t-1}) = f_3(x_t \mid x^*_t).$$

(iii) The irrelevance of lagged variables to the choice d_t given true state x^*_t:

$$f(d_t \mid x^*_t, d_{t-1}, x^*_{t-1}, x_{t-1}) = f_1(d_t \mid x^*_t).$$

(iv) The irrelevance of proxy x_t to the state transition:

$$f(x^*_t \mid d_{t-1}, x^*_{t-1}, x_{t-1}) = f_2(x^*_t \mid d_{t-1}, x^*_{t-1}).$$

https://doi.org/10.1017/S0266466617000081 Published online by Cambridge University Press

It is not difficult to see that, together, (i)–(iv) are equivalent to Assumption 2. Part (i) is standard in the dynamic discrete choice literature and we are not adding any new assumption here. Likewise, parts (iii) and (iv) are intuitive in the spirit of dynamic discrete choice models. As such, part (ii) is effectively the only new condition we are invoking in Assumption 2 compared to the existing literature. It is satisfied if the measurement error, defined as the difference between proxy x_t and true state x_t^*, is independent of $(d_t, x_t^*, d_{t-1}, x_{t-1}^*, x_{t-1})$. This classical error assumption can be restrictive for some applications, and we, therefore, present a way to relax this assumption in Appendix A.2.

In the context of parental investment in a child's abilities and skills, part (i) means that the investment decision d_t is based only on the current actual abilities x_t^*, and the past information (d_{t-1}, x_{t-1}^*) is irrelevant to this decision. This setup is consistent with the investment model of Cunha et al. (2010; equation 4.2) motivated by economic theory. Part (ii) means that the PIAT score x_t only reflects on current actual abilities x_t^*, and past information $(d_{t-1}, x_{t-1}^*, x_{t-1})$ as well as the current investment decision d_t are irrelevant to the current test score x_t once the current abilities x_t^* are controlled for. This measurement feature is consistent with Cunha et al. (2010). The implication of part (iii) is similar to that of part (i): the investment decision d_t is based only on the current actual abilities x_t^*, and the test score x_t as well as the past information (d_{t-1}, x_{t-1}^*) are irrelevant to this decision. Again, this is consistent with the investment model of Cunha et al. (2010; Equation 4.2). Part (iv) means that the dynamics of human capital x_t^* depends on the parental investment d_{t-1} and the human capital stock x_{t-1}^* in the last period, and does not depend on the measurement x_{t-1} of the actual stock x_{t-1}^*. This setup is also consistent with the skill production function of Cunha et al. (2010; equation 2.1).

Because the state variable x_t^* of interest is unit-less and unobserved, we require a restriction of location- and scale-normalization. To this goal, the transition rule for the unobserved state variable and the state-proxy relation are semiparametrically specified, as follows.

Assumption 3 (Semi-parametric restrictions on the unobservables)**.** The transition rule for the unobserved state variable and the state-proxy relation are semiparametrically specified by

$$f_2\left(x_t^* | d_{t-1}, x_{t-1}^*\right): \qquad x_t^* = \alpha^d + \gamma^d x_{t-1}^* + \eta_t^d \qquad \text{if } d_{t-1} = d \tag{2.1}$$

$$f_3\left(x_t | x_t^*\right): \qquad x_t = x_t^* + \varepsilon_t, \tag{2.2}$$

where ε_t and η_t^d have mean zero for each d, and satisfy

$$\varepsilon_t \perp\!\!\!\perp (\{d_\tau\}_\tau, \{x_\tau^*\}_\tau, \{\varepsilon_\tau\}_{\tau \neq t}) \qquad \text{for all } t$$

$$\eta_t^d \perp\!\!\!\perp \left(\{d_\tau\}_{\tau < t}, \{x_\tau^*\}_{\tau < t}\right) \qquad \text{for all } t.$$

When we consider the discrete choice d_t of an investment decision, for example, it is important that the coefficients, (α^d, γ^d), are allowed to depend on the amount d of investments as how much is invested will likely affect the dynamics

https://doi.org/10.1017/S0266466617000081 Published online by Cambridge University Press

of technological evolution. For example, ordering such as $\gamma^0 < \gamma^1 < \cdots < \gamma^{\bar{d}}$ allows for the dynamic complementarity of parental investment in human capital stocks in the framework of Cunha et al. (2010). As such, we allow these parameters to have the d superscripts in (2.1). The semiparametric model (2.2) of the state-proxy relation specifies the proxy x_t as a measurement of the latent technology x_t^* with a classical error. As it is often restrictive in applications, we also discuss how to relax this classical-error assumption in Section A.2.

By Assumption 3, closed-form identification of the transition rule for x_t^* and the proxy model for x_t^* follows from identification of the parameters (α^d, γ^d) for each d and from identification of the nonparametric distributions of the unobservables, ε_t, x_t^*, and η_t^d for each d. We show that identification of the parameters (α^d, γ^d) follows from the empirically testable rank condition stated as Assumption 4 below.[1] We also obtain identification of the nonparametric distributions of the unobservables, ε_t, x_t^*, and η_t^d, by deconvolution methods under the regularity condition stated as Assumption 5 below.

Assumption 4 (Testable rank condition)**.** $\Pr(d_{t-1} = d) > 0$ and the following matrix is nonsingular for each d.

$$\begin{bmatrix} 1 & \mathrm{E}[x_{t-1} \mid d_{t-1} = d] \\ \mathrm{E}[d_{t-2} \mid d_{t-1} = d] & \mathrm{E}[x_{t-1} d_{t-2} \mid d_{t-1} = d] \end{bmatrix}$$

Assumption 4 is empirically testable, as is the common rank condition in generic econometric contexts. While we propose a simple affine model in (2.1) for Assumption 3, we remark that this particular functional form is not crucial for our identification result. We may include higher-order terms (and interaction terms, which will be relevant in Appendix A.1 where an observed state variable is included) as far as the corresponding rank condition, analogously to the one in Assumption 4, is satisfied. In that case, higher-order lags of d_t will be needed to meet the rank condition of larger dimensions. Finally, we use the following regularity conditions.

Assumption 5 (Regularity)**.** The random variable x_t^* has a bounded conditional first moment given d_t. The conditional characteristic function of x_t^* given $d_t = d$ does not vanish on the real line, and is absolutely integrable. Random variables ε_t and η_t^d have bounded first moments and have absolutely integrable characteristic functions that do not vanish on the real line.

Assumption 5 is satisfied by common distribution families, such as the normal family, and is standard in the deconvolution literature.

2.2. The Result

Under the five assumptions stated and discussed in Section 2.1, we obtain the following closed-form identification result for the three components of the Markov kernel.

THEOREM 1. *If Assumptions 1, 2, 3, 4, and 5 are satisfied, then the three components,* $f_1(d_t|x_t^*)$, $f_2(x_t^*|d_{t-1},x_{t-1}^*)$, $f_3(x_t|x_t^*)$, *of the Markov kernel* $f(d_t,x_t^*,x_t|d_{t-1},x_{t-1}^*,x_{t-1})$ *are identified with closed-form formulas.*

Proof. Our closed-form identification involves three steps.

Step 1: Closed-Form Identification of the Transition Rule $f_2(x_t^*|d_{t-1},x_{t-1}^*)$**:** First, we show the identification of the parameters and the distributions in the transition law of x_t^*. As

$$\begin{aligned} x_t = x_t^* + \varepsilon_t &= \sum_d \mathbb{1}\{d_{t-1}=d\}[\alpha^d + \gamma^d x_{t-1}^* + \eta_t^d] + \varepsilon_t \\ &= \sum_d \mathbb{1}\{d_{t-1}=d\}[\alpha^d + \gamma^d x_{t-1} + \eta_t^d - \gamma^d \varepsilon_{t-1}] + \varepsilon_t \end{aligned}$$

is true under Assumption 3, we obtain the following equalities for each d:

$$\begin{aligned} \mathrm{E}[x_t \mid d_{t-1}=d] &= \alpha^d + \gamma^d \mathrm{E}[x_{t-1} \mid d_{t-1}=d] \\ &\quad - \mathrm{E}[\gamma^d \varepsilon_{t-1} \mid d_{t-1}=d] + \mathrm{E}[\eta_t^d \mid d_{t-1}=d] + \mathrm{E}[\varepsilon_t \mid d_{t-1}=d] \\ &= \alpha^d + \gamma^d \mathrm{E}[x_{t-1} \mid d_{t-1}=d] \end{aligned}$$

$$\begin{aligned} \mathrm{E}[x_t d_{t-2} \mid d_{t-1}=d] &= \alpha^d \mathrm{E}[d_{t-2} \mid d_{t-1}=d] + \gamma^d \mathrm{E}[x_{t-1}d_{t-2} \mid d_{t-1}=d] \\ &\quad - \mathrm{E}[\gamma^d \varepsilon_{t-1} d_{t-2} \mid d_{t-1}=d] + \mathrm{E}[\eta_t^d d_{t-2} \mid d_{t-1}=d] \\ &\quad + \mathrm{E}[\varepsilon_t d_{t-2} \mid d_{t-1}=d] \\ &= \alpha^d \mathrm{E}[d_{t-2} \mid d_{t-1}=d] + \gamma^d \mathrm{E}[x_{t-1}d_{t-2} \mid d_{t-1}=d]. \end{aligned}$$

The independence and zero mean assumptions for η_t^d and ε_t stated in Assumption 3 are used above. We, thus, obtain the linear equation

$$\begin{bmatrix} \mathrm{E}[x_t \mid d_{t-1}=d] \\ \mathrm{E}[x_t d_{t-2} \mid d_{t-1}=d] \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{E}[x_{t-1} \mid d_{t-1}=d] \\ \mathrm{E}[d_{t-2} \mid d_{t-1}=d] & \mathrm{E}[x_{t-1}d_{t-2} \mid d_{t-1}=d] \end{bmatrix} \begin{bmatrix} \alpha^d \\ \gamma^d \end{bmatrix}.$$

By the nonsingularity of the matrix on the right-hand side stated in Assumption 4, we can identify the parameters (α^d, γ^d) by

$$\begin{bmatrix} \alpha^d \\ \gamma^d \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{E}[x_{t-1} \mid d_{t-1}=d] \\ \mathrm{E}[d_{t-2} \mid d_{t-1}=d] & \mathrm{E}[x_{t-1}d_{t-2} \mid d_{t-1}=d] \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{E}[x_t \mid d_{t-1}=d] \\ \mathrm{E}[x_t d_{t-2} \mid d_{t-1}=d] \end{bmatrix}.$$

Next, we show the identification of the distributions of ε_t and η_t^d for each d. Observe that

$$\begin{aligned} &\mathrm{E}\left[\exp(is_1 x_{t-1} + is_2 x_t)\,|d_{t-1}=d\right] \\ &\quad = \mathrm{E}\left[\exp\left(is_1\left(x_{t-1}^* + \varepsilon_{t-1}\right) + is_2\left(\alpha^d + \gamma^d x_{t-1}^* + \eta_t^d + \varepsilon_t\right)\right)|d_{t-1}=d\right] \\ &\quad = \mathrm{E}\left[\exp\left(i\left(s_1 x_{t-1}^* + s_2\alpha^d + s_2\gamma^d x_{t-1}^*\right)\right)|d_{t-1}=d\right]\mathrm{E}\left[\exp\left(is_1\varepsilon_{t-1}\right)\right]\mathrm{E}\left[\exp\left(is_2\left(\eta_t^d + \varepsilon_t\right)\right)\right] \end{aligned}$$

follows from the independence assumptions for η_t^d and ε_t stated in Assumption 3. Taking the derivative with respect to s_2 yields

$$\left[\frac{\partial}{\partial s_2}\ln \mathrm{E}\left[\exp\left(is_1x_{t-1}+is_2x_t\right)\left|d_{t-1}=d\right.\right]\right]_{s_2=0}=\frac{\mathrm{E}\left[i\left(\alpha^d+\gamma^d x_{t-1}^*\right)\exp\left(is_1x_{t-1}^*\right)\left|d_{t-1}=d\right.\right]}{\mathrm{E}\left[\exp\left(is_1x_{t-1}^*\right)\left|d_{t-1}=d\right.\right]}$$
$$=i\alpha^d+\gamma^d\frac{\partial}{\partial s_1}\ln \mathrm{E}\left[\exp\left(is_1x_{t-1}^*\right)\left|d_{t-1}=d\right.\right],$$

where the switch of the differential and integral operators is permissible provided there exists $h \in L^1(F_{x_{t-1}^*|d_{t-1}=d})$ such that $\left|i\left(\alpha^d+\gamma^d x_{t-1}^*\right)\exp\left(is_1x_{t-1}^*\right)\right| < h(x_{t-1}^*)$ holds for all x_{t-1}^*, which follows from the bounded conditional moment condition provided in Assumption 5, and the denominator is nonzero as the conditional characteristic function of x_t^* given d_t does not vanish on the real line under Assumption 5. Therefore, we have

$$\mathrm{E}\left[\exp\left(isx_{t-1}^*\right)\left|d_{t-1}=d\right.\right]=\exp\left[\int_0^s\frac{\mathrm{E}\left[i\left(x_t-\alpha^d\right)\exp\left(is_1x_{t-1}\right)\left|d_{t-1}=d\right.\right]}{\gamma^d\mathrm{E}\left[\exp\left(is_1x_{t-1}\right)\left|d_{t-1}=d\right.\right]}ds_1\right].$$

On the other hand, from the proxy model and the independence conditions for ε_t stated in Assumption 3, we also have

$$\mathrm{E}\left[\exp\left(isx_{t-1}\right)\left|d_{t-1}=d\right.\right]=\mathrm{E}\left[\exp\left(isx_{t-1}^*\right)\left|d_{t-1}=d\right.\right]\mathrm{E}\left[\exp\left(is\varepsilon_{t-1}\right)\right].$$

Combining the above two equations, we obtain the following identifying formula using any d.

$$\mathrm{E}\left[\exp\left(is\varepsilon_{t-1}\right)\right]=\frac{\mathrm{E}\left[\exp\left(isx_{t-1}\right)|d_{t-1}=d\right]}{\mathrm{E}\left[\exp\left(isx_{t-1}^*\right)\left|d_{t-1}=d\right.\right]}=\frac{\mathrm{E}\left[\exp\left(isx_{t-1}\right)|d_{t-1}=d\right]}{\exp\left[\int_0^s\frac{\mathrm{E}\left[i\left(x_t-\alpha^d\right)\exp\left(is_1x_{t-1}\right)|d_{t-1}=d\right]}{\gamma^d\mathrm{E}\left[\exp\left(is_1x_{t-1}\right)|d_{t-1}=d\right]}ds_1\right]}.$$

This argument holds for all t, so that we can identify the characteristic function of ε_t by

$$\phi_{\varepsilon_t}(s)=\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]=\frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_t=d\right]}{\exp\left[\int_0^s\frac{\mathrm{E}\left[i\left(x_{t+1}-\alpha^d\right)\exp\left(is_1x_t\right)|d_t=d\right]}{\gamma^d\mathrm{E}\left[\exp\left(is_1x_t\right)|d_t=d\right]}ds_1\right]} \tag{2.3}$$

using any d.

To identify the distribution of η_t^d for each d, consider

$$x_t+\gamma^d\varepsilon_{t-1}=\alpha^d+\gamma^d x_{t-1}+\varepsilon_t+\eta^d$$

which holds under Assumption 3. From this equality,

$$\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d\right]\mathrm{E}\left[\exp\left(is\gamma^d\varepsilon_{t-1}\right)\right]=\mathrm{E}\left[\exp\left(is\left(\alpha^d+\gamma^d x_{t-1}\right)\right)\left|d_{t-1}=d\right.\right]$$
$$\times\mathrm{E}\left[\exp\left(is\eta_t^d\right)\right]\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]$$

follows by the independence assumptions for η_t^d and ε_t stated in Assumption 3. Therefore, by the identifying formula (2.3) for ϕ_{ε_t}, the characteristic function of η_t^d can be expressed by

$$\phi_{\eta_t^d}(s)=\mathrm{E}\left[\exp\left(is\eta_t^d\right)\right]=\frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d\right]\cdot\mathrm{E}\left[\exp\left(is\gamma^d\varepsilon_{t-1}\right)\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d+\gamma^d x_{t-1})\right)\left|d_{t-1}=d\right.\right]\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]}$$
$$=\frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d\right]\cdot\exp\left[\int_0^s\frac{\mathrm{E}\left[i\left(x_{t+1}-\alpha^d\right)\exp\left(is_1x_t\right)|d_t=d\right]}{\gamma^d\mathrm{E}\left[\exp\left(is_1x_t\right)|d_t=d\right]}ds_1\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d+\gamma^d x_{t-1})\right)\left|d_{t-1}=d\right.\right]\cdot\mathrm{E}\left[\exp\left(isx_t\right)\left|d_t=d\right.\right]}$$
$$\times\frac{\mathrm{E}\left[\exp\left(is\gamma^d x_{t-1}\right)\left|d_{t-1}=d\right.\right]}{\exp\left[\int_0^{s\gamma^d}\frac{\mathrm{E}\left[i\left(x_t-\alpha^d\right)\exp\left(is_1x_{t-1}\right)|d_{t-1}=d\right]}{\gamma^d\mathrm{E}\left[\exp\left(is_1x_{t-1}\right)|d_{t-1}=d\right]}ds_1\right]}. \tag{2.4}$$

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The denominator on the right-hand side is nonzero, as the conditional and unconditional characteristic functions do not vanish on the real line under Assumption 5. Letting $\mathcal{F}$ denote the Fourier transform operator defined by

$$(\mathcal{F}\phi)(\xi) = \frac{1}{2\pi}\int e^{-is\xi}\phi(s)ds \qquad \text{for all } \phi \in L^1(\mathbb{R}) \text{ and } \xi \in \mathbb{R},$$

we identify $f_{\eta_t^d}$ by

$$f_{\eta_t^d}(\eta) = \left(\mathcal{F}\phi_{\eta_t^d}\right)(\eta) \qquad \text{for all } \eta,$$

under Assumption 5, where the characteristic function $\phi_{\eta_t^d}$ is identified in (2.4). We can use this identified density function $f_{\eta_t^d}$ to identify the transition rule $f_2\left(x_t^* \middle| d_{t-1}, x_{t-1}^*\right)$ with

$$f_2\left(x_t^* \middle| d_{t-1}, x_{t-1}^*\right) = \sum_d \mathbb{1}\{d_{t-1} = d\} f_{\eta_t^d}\left(x_t^* - \alpha^d - \gamma^d x_{t-1}^*\right).$$

In summary, we obtain the closed-form identifying formula for the law of state transition $f_2\left(x_t^* \middle| d_{t-1}, x_{t-1}^*\right)$:

$$f_2\left(x_t^* \middle| d_{t-1}, x_{t-1}^*\right) = \sum_d \frac{\mathbb{1}\{d_{t-1} = d\}}{2\pi}\int \exp\left(-is\left(x_t^* - \alpha^d - \gamma^d x_{t-1}^*\right)\right)$$
$$\times \frac{\mathrm{E}\left[\exp(isx_t)\,|d_{t-1} = d\right] \cdot \exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1} - \alpha^{d'})\exp(is_1 x_t)|d_t = d'\right]}{\gamma^{d'}\mathrm{E}\left[\exp(is_1 x_t)|d_t = d'\right]}ds_1\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d + \gamma^d x_{t-1})\right)|d_{t-1} = d\right] \cdot \mathrm{E}\left[\exp(isx_t)\,|d_t = d\right]}$$
$$\times \frac{\mathrm{E}\left[\exp\left(is\gamma^d x_{t-1}\right)\middle|d_{t-1} = d'\right]}{\exp\left[\int_0^{s\gamma^d} \frac{\mathrm{E}\left[i(x_t - \alpha^{d'})\exp\left(is_1 x_{t-1}\right)|d_{t-1} = d'\right]}{\gamma^{d'}\mathrm{E}\left[\exp\left(is_1 x_{t-1}\right)|d_{t-1} = d'\right]}ds_1\right]}\, ds.$$

using any d'. This completes Step 1.

Step 2: Closed-Form Identification of the Proxy Model $f_3\left(x_t \mid x_t^*\right)$: We can write the density function of ε_t by

$$f_{\varepsilon_t}(\varepsilon) = \left(\mathcal{F}\phi_{\varepsilon_t}\right)(\varepsilon) \qquad \text{for all } \varepsilon,$$

where the characteristic function ϕ_{ε_t} is identified in (2.3) with a closed-form formula. Provided this identified density function f_{ε_t}, we identify the proxy model $f_3(x_t \mid x_t^*)$ by

$$f_3(x_t \mid x_t^*) = f_{\varepsilon_t}(x_t - x_t^*).$$

In summary, we obtain the closed-form identifying formula for the proxy model $f_3(x_t \mid x_t^*)$:

$$f_3(x_t \mid x_t^*) = \frac{1}{2\pi}\int \frac{\exp\left(-is(x_t - x_t^*)\right) \cdot \mathrm{E}\left[\exp(isx_t)\,|d_t = d\right]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1} - \alpha^d)\exp(is_1 x_t)|d_t = d\right]}{\gamma^d \mathrm{E}\left[\exp(is_1 x_t)|d_t = d\right]}ds_1\right]}ds$$

using any d. This completes Step 2.

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Step 3: Closed-Form Identification of the CCP $f_1\left(d_t|x_t^*\right)$**:** We can write

$$\begin{aligned}
\mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(isx_t\right)\right] &= \mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(isx_t^*+is\varepsilon_t\right)\right] \\
&= \mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(isx_t^*\right)\right]\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[\mathbb{1}\{d_t=d\}|x_t^*\right]\exp\left(isx_t^*\right)\right]\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]
\end{aligned}$$

by the independence assumption for ε_t stated in Assumption 3 and the law of iterated expectations. Therefore, we obtain

$$\begin{aligned}
\frac{\mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(isx_t\right)\right]}{\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]} &= \mathrm{E}\left[\mathrm{E}\left[\mathbb{1}\{d_t=d\}|x_t^*\right]\exp\left(isx_t^*\right)\right] \\
&= \int \exp\left(isx^*\right)\mathrm{E}\left[\mathbb{1}\{d_t=d\}|x_t^*=x^*\right]f_{x_t^*}\left(x^*\right)dx^*.
\end{aligned}$$

This is the Fourier inversion of $\mathrm{E}\left[\mathbb{1}\{d_t=d\}|x_t^*=\cdot\ \right]f_{x_t^*}(\cdot)$. On the other hand, the Fourier inversion of $f_{x_t^*}$ can be found as

$$\mathrm{E}\left[\exp\left(isx_t^*\right)\right]=\frac{\mathrm{E}\left[\exp\left(isx_t\right)\right]}{\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]}.$$

Therefore, we find the closed-form expression for CCP $f_1\left(d_t|x_t^*\right)$ as follows.

$$\Pr\left(d_t=d|x_t^*\right)=\mathrm{E}\left[\mathbb{1}\{d_t=d\}|x_t^*\right]=\frac{\mathrm{E}\left[\mathbb{1}\{d_t=d\}|x_t^*\right]f_{x_t^*}\left(x_t^*\right)}{f_{x_t^*}\left(x_t^*\right)}=\frac{\left(\mathcal{F}\phi_{(d)x_t^*}\right)\left(x_t^*\right)}{\left(\mathcal{F}\phi_{x_t^*}\right)\left(x_t^*\right)},$$

where the ‘phi’ functions in the last expression are

$$\begin{aligned}
\phi_{(d)x_t^*}(s) &= \frac{\mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(isx_t\right)\right]}{\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]} \\
&= \frac{\mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(isx_t\right)\right]\cdot\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{d'})\exp(is_1x_t)|d_t=d'\right]}{\gamma^{d'}\mathrm{E}\left[\exp(is_1x_t)|d_t=d'\right]}ds_1\right]}{\mathrm{E}\left[\exp\left(isx_t\right)|d_t=d'\right]}
\end{aligned}$$

and

$$\phi_{x_t^*}(s)=\frac{\mathrm{E}\left[\exp\left(isx_t\right)\right]}{\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]}=\frac{\mathrm{E}\left[\exp\left(isx_t\right)\right]\cdot\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{d'})\exp(is_1x_t)|d_t=d'\right]}{\gamma^{d'}\mathrm{E}\left[\exp(is_1x_t)|d_t=d'\right]}ds_1\right]}{\mathrm{E}\left[\exp\left(isx_t\right)|d_t=d'\right]}$$

from (2.3) using any d'. In summary, we obtain the closed-form identifying formula for the CCP $f_1(d_t\mid x_t^*)$:

$$\begin{aligned}
\Pr\left(d_t=d|x_t^*\right) &= \frac{\left(\mathcal{F}\phi_{(d)x_t^*}\right)\left(x_t^*\right)}{\left(\mathcal{F}\phi_{x_t^*}\right)\left(x_t^*\right)} \\
&= \int \exp\left(-isx_t^*\right)\cdot\mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(isx_t\right)\right]\frac{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{d'})\exp(is_1x_t)|d_t=d'\right]}{\gamma^{d'}\mathrm{E}\left[\exp(is_1x_t)|d_t=d'\right]}ds_1\right]}{\mathrm{E}\left[\exp\left(isx_t\right)|d_t=d'\right]}ds \Bigg/ \\
&\quad \int \exp\left(-isx_t^*\right)\cdot\mathrm{E}\left[\exp\left(isx_t\right)\right]\frac{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{d'})\exp(is_1x_t)|d_t=d'\right]}{\gamma^{d'}\mathrm{E}\left[\exp(is_1x_t)|d_t=d'\right]}ds_1\right]}{\mathrm{E}\left[\exp\left(isx_t\right)|d_t=d'\right]}ds
\end{aligned}$$

using any d'. This completes Step 3. ∎

2.3. Closed-Form Analog Estimation

By the closed-form identifying formulas for the three components, $f_1\left(d_t|x_t^*\right)$, $f_2\left(x_t^*|d_{t-1},x_{t-1}^*\right)$ and $f_3\left(x_t|x_t^*\right)$, obtained in the theorem above, it is straightforward to develop sample-analog closed-form estimators of them. First, an analog estimator for the CCP $f_1(d_t \mid x_t^*)$ can be written as

$$\widehat{f_1}(d \mid x^*) = \frac{\sum_{j=1}^N \mathbb{1}\{d_{j,t} = d\}\widehat{K}_N((x^* - x_{jt})/h_N)}{\sum_{j=1}^N \widehat{K}_N((x^* - x_{jt})/h_N)}, \tag{2.5}$$

where the estimated deconvoluting kernel $\widehat{K}_N$ takes the form of

$$\widehat{K}_N(x) = \frac{1}{2\pi}\int e^{-isx}\frac{\phi_K(s)}{\widehat{\phi}_{\varepsilon_t}(s/h_N)}ds$$

$$\widehat{\phi}_{\varepsilon_t}(s) = \frac{\widehat{\mathrm{E}}_N\left[\exp(isx_t)\,|d_t = d'\right]}{\exp\left[\int_0^s \frac{\widehat{\mathrm{E}}_N\left[i(x_{t+1}-\widehat{\alpha}^{d'})\exp(is_1x_t)|d_t=d'\right]}{\widehat{\gamma}^{d'}\widehat{\mathrm{E}}_N\left[\exp(is_1x_t)|d_t=d'\right]}ds_1\right]}$$

for any d', with h_N denoting a bandwidth parameter, ϕ_K denoting the Fourier inverse of a kernel function K, and $\widehat{\mathrm{E}}_N$ denoting the sample conditional mean operator. It is important to note that $\widehat{\mathrm{E}}_N$ converges at the parametric rate under standard conditions because of the discreteness of d_t. The Nadaraya–Watson-type estimator (2.5) with the estimated deconvoluting kernel $\widehat{K}_N$ is analyzed in the literature, and consistency results are available (e.g., Schennach, 2004b). Nonparametric convergence rates depend on smoothness assumptions for the distributions of x_t^* and ε_t.

Similarly, the remaining two components of the Markov kernel can be written as deconvoluting kernel density estimators. Let

$$\widehat{\phi}_{\eta_t^d}(s) = \frac{\widehat{\mathrm{E}}_N[\exp(isx_t) \mid d_{t-1} = d]\cdot\widehat{\phi}_{\varepsilon_{t-1}}(\widehat{\gamma}^d s)}{\widehat{\mathrm{E}}_N[\exp(is(\widehat{\alpha}^d + \widehat{\gamma}^d x_{t-1})) \mid d_{t-1} = d]\cdot\widehat{\phi}_{\varepsilon_t}(s)}$$

estimate the characteristic function of η_t^d. The transition rule $f_2(x_t^* \mid d_{t-1}, x_{t-1}^*)$ and the proxy model $f_3(x_t \mid x_t^*)$ can be estimated by

$$\widehat{f_2}(x^* \mid d, x_-^*) = \frac{1}{2\pi}\int e^{-is(x^*-\widehat{\alpha}^d-\widehat{\gamma}^d x_-^*)}\phi_K(sh_N)\widehat{\phi}_{\eta_t^d}(s)ds \qquad \text{and}$$

$$\widehat{f_3}(x \mid x^*) = \frac{1}{2\pi}\int e^{-is(x-x^*)}\phi_K(sh_N)\widehat{\phi}_{\varepsilon_t}(s)ds,$$

respectively. The consistency of $\widehat{f_3}$ can be shown by directly applying Li and Vuong (1998). On the other hand, the consistency of $\widehat{f_2}$ does not directly follow from Li and Vuong or any other method to our best knowledge. Therefore, we discuss asymptotic analysis of this estimator in Section C in supplementary material to this article, available at Cambridge Journals Online (journals.cambridge.org/ect). Similarly to $\widehat{f_1}$, nonparametric rates of convergence for these two estimators depend on smoothness assumptions for the distributions of x_t^*, ε_t, and η_t^d.

Some remarks are in order concerning estimation and inference. In the literature on deconvoluting kernel estimation, inference results are not yet well established. For cases of known error densities, some papers in statistics certainly develop methods of inference (e.g., Bissantz, Dümbgen, Holtzmann, and Munk, 2007). However, for cases of unknown error densities, as in our model, the existing literature focuses only on consistency, and does not provide methods of inference.[2] Inference for this class of estimators is a nontrivial issue even under very simple settings. Hence, some future advancements in this field are awaited. Second, given the nonparametric identification result stated in Theorem 1, one could alternatively use the maximum likelihood estimator as Hu and Schennach (2008) did in a related nonparametric context. However, such an alternative approach may suffer from heavy computational burdens and practical difficulties in finding a global solution. On the other hand, our closed-form estimator requires less computation, with one-dimensional numerical integration being the most demanding computational part, and is guaranteed to find a global solution unlike extremal estimation over large dimensions.

3. IDENTIFICATION OF DYNAMIC DISCRETE CHOICE MODELS

Once the two Markov components, the CCP $f_1(d_t \mid x_t^*)$ and the law of state transition $f_2(x_t^* \mid d_{t-1}, x_{t-1}^*)$, are identified, one can apply one of the existing methods to further identify the underlying structure of dynamic discrete choice models in the framework of Hotz and Miller (1993) and Hotz et al. (1994). This section briefly explains how existing methods can be applied in our context, borrowing a state-of-the-art technique (e.g., Pesendorfer and Schmidt-Dengler, 2008; Srisuma and Linton, 2012; Sanches, Silva, and Srisuma, 2015; Srisuma 2015).

Suppose that an agent receives the θ-dependent current-time payoff

$$U_\theta(d_t, x_t^*, \upsilon_t) = \pi_\theta(d_t, x_t^*) + \sum_{d=0}^{\bar{d}} \upsilon_{dt} \mathbb{1}\{d_t = d\}$$

at time t if she makes the choice $d_t = d$ under the state (x_t^*, υ_t), where $\upsilon_t = \{\upsilon_{dt}\}_{d=0}^{\bar{d}}$ and υ_{dt} is a private payoff shock associated with the choice $d_t = d$ at time t, independently following the type I extreme value distribution. The dynamically optimizing agent sequentially makes decisions $\{d_t\}$ to maximize the expected discounted sum of payoffs

$$\mathrm{E}\left[\sum_{s=t}^{\infty} \rho^{s-t} U_\theta(d_t, x_t^*, \upsilon_t) \middle| x_t^*, \upsilon_t\right],$$

where $\rho \in (0,1)$ is the rate of time preference. Let V_θ denote the value function that follows from this optimal choice rule. Economists are interested in consistently estimating the structural parameters θ. The rate ρ of time preference is assumed to be known.[3]

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We outline the procedure to identify and estimate the structural parameters θ following Srisuma and Linton (2012) and Srisuma (2015). First, given the CCP f_1 identified in Section 2, identify $r_\theta = \mathrm{E}[U_\theta(d_t, x_t^*, \upsilon_t) \mid x_t^* = \cdot\,]$ by

$$r_\theta(x^*) = \sum_{d=0}^{\bar{d}} f_1(d \mid x^*)\pi_\theta(d, x^*) + \chi + \sum_{d=0}^{\bar{d}} f_1(d \mid x^*) \log f_1(d \mid x^*), \tag{3.1}$$

where $\chi \approx 0.577$ is the Euler's constant. Second, given the CCP f_1 and the law of state transition f_2 identified in Section 2, identify the conditional expectation operators $\mathcal{L}$ and $\mathcal{H}$

$$\mathcal{L}g(x^*) = \int g(x^{**}) \sum_{d=0}^{\bar{d}} f_1(d \mid x^*) f_2(dx^{**} \mid d, x^*) \qquad \text{and} \tag{3.2}$$

$$Hg(d, x^*) = \int g(x^{**}) f_2(dx^{**} \mid d, x^*). \tag{3.3}$$

Third, given r_θ and $\mathcal{L}$ identified in (3.1) and (3.2), identify $m_\theta = \mathrm{E}[V_\theta(x_t^*, \upsilon_t) \mid x_t^* = \cdot\,]$ by

$$m_\theta = (I - \rho L)^{-1} r_\theta. \tag{3.4}$$

Fourth, given H and m_θ identified in (3.3) and (3.4), identify the state action value function

$$\upsilon_\theta = \pi_\theta + \rho H m_\theta. \tag{3.5}$$

Finally, given υ_θ identified in (3.5), obtain the logit conditional likelihood function

$$P_\theta(d \mid x^*) = \exp(\upsilon_\theta(d, x^*)) \Bigg/ \sum_{d=0}^{\bar{d}} \exp(\upsilon_\theta(d', x^*)).$$

An analog estimator of this likelihood function can be used to estimate the parameters θ in the standard manner.

4. SUMMARY

In this paper, we show that the structure of forward-looking agents making a sequence of discrete choices can be identified without observing the true state variable, provided that a proxy for the unobserved state variable is available in the data. Our approach combines econometric methods in the following manner. First, we identify Markov components, including the CCP and the law of state transition, by using a proxy variable. This is done by explicitly solving integral equations based on deconvolution methods. Second, the CCP-based method is applied to the preliminarily identified Markov components to obtain the structural parameters of a current-time payoff.

NOTES

1. This matrix consists of moments estimable at the parametric rate of convergence, and hence the standard rank tests (e.g., Cragg and Donald, 1997; Robin and Smith, 2000; Kleibergen and Paap, 2006) can be used.

2. An exception is a recent paper by Kato and Sasaki (2016). They develop methods of inference for cases of unknown error densities with symmetric error distributions.

3. This rate is generally nonidentifiable together with the payoffs (Rust,1994; Magnac and Thesmar, 2002).

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APPENDIX A

A.1. Extension 1: Independent Covariates

The baseline model presented in Section 2 includes only the unobserved state variable x_t^*. In empirical applications, we may want to include an observed state variable, which we denote by w_t. The current section presents an extension of the basic identification result with this additional feature of the model. Analogously to Assumptions 1, 2, 3, 4, and 5 stated in the context of the baseline model, we make the following assumptions for the current extension.

Assumption A.1 (First-order Markov process)**.** The quadruple $\left\{d_t, w_t, x_t^*, x_t\right\}$ jointly follows a first-order Markov process.

Assumption A.2 (Independence)**.** The Markov kernel can be decomposed as follows.

$$f\left(d_t, w_t, x_t^*, x_t | d_{t-1}, w_{t-1}, x_{t-1}^*, x_{t-1}\right)$$
$$= f_1\left(d_t | w_t, x_t^*\right) f_2\left(w_t | d_{t-1}, w_{t-1}, x_{t-1}^*\right) f_3\left(x_t^* | d_{t-1}, w_{t-1}, x_{t-1}^*\right) f_4\left(x_t | x_t^*\right),$$

where the four components represent

$f_1\left(d_t \mid w_t, x_t^*\right)$	conditional choice probability (CCP);
$f_2\left(w_t \mid d_{t-1}, w_{t-1}, x_{t-1}^*\right)$	transition rule for the observed state variable;
$f_3\left(x_t^* \mid d_{t-1}, w_{t-1}, x_{t-1}^*\right)$	transition rule for the unobserved state variable; and
$f_4\left(x_t \mid x_t^*\right)$	proxy model.

Like Assumption 2 for the baseline model, this independence assumption is key to our closed-form identification results and is better understood in the context of the standard independence assumptions used in the dynamic discrete choice literature. Consider the following five independence conditions.

(i) Rust's conditional independence assumption:

$$f(d_t, x_t^*, w_t \mid d_{t-1}, x_{t-1}^*, w_{t-1}) = f_1(d_t \mid x_t^*, w_t) \cdot f(x_t^*, w_t \mid d_{t-1}, x_{t-1}^*, w_{t-1})$$

(ii) True state x_t^* is a sufficient statistic for proxy x_t:

$$f(x_t \mid d_t, x_t^*, w_t, d_{t-1}, x_{t-1}^*, w_{t-1}, x_{t-1}) = f_4(x_t \mid x_t^*)$$

(iii) The irrelevance of lagged variables to choice d_t given true state (x_t^*, w_t):

$$f(d_t \mid x_t^*, w_t, d_{t-1}, x_{t-1}^*, w_{t-1}, x_{t-1}) = f_1(d_t \mid x_t^*, w_t)$$

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(iv) The irrelevance of proxy x_t to the state transition:

$$f(x_t^*, w_t \mid d_{t-1}, x_{t-1}^*, w_{t-1}, x_{t-1}) = f(x_t^*, w_t \mid d_{t-1}, x_{t-1}^*, w_{t-1})$$

(v) Conditionally independent evolution of true state (x_t^*, w_t):

$$f(x_t^*, w_t \mid d_{t-1}, x_{t-1}^*, w_{t-1}) = f_3(x_t^* \mid d_{t-1}, x_{t-1}^*, w_{t-1}) f_2(w_t \mid d_{t-1}, x_{t-1}^*, w_{t-1}).$$

Parts (i)–(v) are together equivalent to Assumption A.2. As argued in Section 2.1, parts (i), (iii), and (iv) are less objectionable in the spirit of the dynamic discrete choice literature. Parts (ii) and (v) are new conditions that we are invoking in Assumption A.2 compared to the existing literature. Part (ii) is satisfied if the measurement error, as defined as the difference between proxy x_t and true state x_t^*, is independent of $(d_t, x_t^*, w_t, d_{t-1}, x_{t-1}^*, w_{t-1}, x_{t-1})$. We present a way to relax this assumption in Appendix A.2. Part (iv) is satisfied if the two states w_t and x_t^* evolve independently, conditional on the past. This part may also be restrictive in some applications. We discuss how this assumption can be relaxed in Section A.3.

Assumption A.3 (Semi-parametric restrictions on the unobservables)**.** The transition rule for the unobserved state variable and the state-proxy relation are semiparametrically specified as follows:

$$f_3\left(x_t^* | d_{t-1}, w_{t-1}, x_{t-1}^*\right): \qquad x_t^* = \alpha^d + \beta^d w_{t-1} + \gamma^d x_{t-1}^* + \eta_t^d \qquad \text{if } d_{t-1} = d \quad \textbf{(A.1)}$$

$$f_4\left(x_t | x_t^*\right): \qquad x_t = x_t^* + \varepsilon_t, \qquad \textbf{(A.2)}$$

where ε_t and η_t^d have mean zero for each d, and satisfy

$$\begin{array}{ll} \varepsilon_t \perp\!\!\!\perp (\{d_\tau\}_\tau, \{x_\tau^*\}_\tau, \{w_\tau\}_\tau, \{\varepsilon_\tau\}_{\tau \neq t}) & \text{for all } t \\ \eta_t^d \perp\!\!\!\perp \left(d_\tau, x_\tau^*, w_\tau\right) & \text{for all } \tau < t \text{ for all } t. \end{array}$$

Assumption A.4 (Testable rank condition)**.** $\Pr(d_{t-1} = d) > 0$ and the following matrix is nonsingular for each d.

$$\begin{bmatrix} 1 & \mathrm{E}[w_{t-1} \mid d_{t-1} = d] & \mathrm{E}[x_{t-1} \mid d_{t-1} = d] \\ \mathrm{E}[w_{t-1} \mid d_{t-1} = d] & \mathrm{E}[w_{t-1}^2 \mid d_{t-1} = d] & \mathrm{E}[x_{t-1} w_{t-1} \mid d_{t-1} = d] \\ \mathrm{E}[w_t \mid d_{t-1} = d] & \mathrm{E}[w_{t-1} w_t \mid d_{t-1} = d] & \mathrm{E}[x_{t-1} w_t \mid d_{t-1} = d] \end{bmatrix}$$

Like Assumption 4, this assumption is empirically testable. While we propose a simple affine model in (A.1) for Assumption A.3, this particular functional form is not crucial to our identification result. We may include arbitrary higher-order terms and interaction terms of w_{t-1} and x_{t-1}^* as far as the corresponding rank condition, analogous to the one in Assumption A.4, is satisfied. In that case, powers and/or higher-order lags of w_t will be needed to meet the rank condition of larger dimensions. Finally, we use the following regularity conditions.

Assumption A.5 (Regularity)**.** The random variables w_t and x_t^* have bounded conditional first moments given d_t. The conditional characteristic functions of w_t and x_t^* given $d_t = d$ do not vanish on the real line, and are absolutely integrable. The conditional characteristic function of (x_{t-1}^*, w_t) given (d_{t-1}, w_{t-1}) and the conditional characteristic function of x_t^* given w_t are absolutely integrable. Random variables ε_t and η_t^d have bounded first moments and absolutely integrable characteristic functions that do not vanish on the real line.

Under this list of five assumptions, we obtain the following closed-form identification result for the four components of the Markov kernel.

THEOREM A.1. *If Assumptions A.1, A.2, A.3, A.4, and A.5 are satisfied, then the four components* $f_1(d_t|w_t,x_t^*)$, $f_2(w_t|d_{t-1},w_{t-1},x_{t-1}^*)$, $f_3(x_t^*|d_{t-1},w_{t-1},x_{t-1}^*)$, $f_4(x_t|x_t^*)$ *of the Markov kernel*

$$f(d_t,w_t,x_t^*,x_t|d_{t-1},w_{t-1},x_{t-1}^*,x_{t-1})$$

are identified with closed-form formulas.

Proof is provided in Section B.1 in supplementary material to this article, available at Cambridge Journals Online (journals.cambridge.org/ect).

A.2. Extension 2: Relaxing the Classical Error Assumption

The models presented in Sections 2 and A.1 assume classical measurement errors. To relax this assumption, we now allow the relationship between the proxy and the unobserved state variable to depend on the endogenous choice made in the previous period. This generalization is useful if the past action can affect the measurement nature of the proxy variable. For example, when the choice d_t leads to entry and exit status of a firm, what proxy measure we may obtain for the unobserved productivity of the firm may differ depending on whether the firm is inside or outside the market.

To allow the proxy model to depend on endogenous actions, we modify Assumptions A.2, A.3, A.4, and A.5 as follows.

Assumption A.2′. The Markov kernel can be decomposed as follows.

$$\begin{aligned} &f(d_t,w_t,x_t^*,x_t|d_{t-1},w_{t-1},x_{t-1}^*,x_{t-1}) \\ &\quad = f_1(d_t|w_t,x_t^*)\, f_2(w_t|d_{t-1},w_{t-1},x_{t-1}^*)\, f_3(x_t^*|d_{t-1},w_{t-1},x_{t-1}^*)\, f_4(x_t|d_{t-1},x_t^*), \end{aligned}$$

where the proxy model now depends on the endogenous choice d_{t-1} made in the last period.

Assumption A.3′. The transition rule for the unobserved state variable and the state-proxy relation are semiparametrically specified by

$$\begin{aligned} f_3(x_t^*|d_{t-1},w_{t-1},x_{t-1}^*):&\qquad x_t^* = \alpha^d + \beta^d w_{t-1} + \gamma^d x_{t-1}^* + \eta_t^d \qquad \text{if } d_{t-1}=d \\ f_4(x_t|d_{t-1},x_t^*):&\qquad x_t = \delta^d x_t^* + \varepsilon_t^d \qquad \text{if } d_{t-1}=d, \end{aligned}$$

where ε_t and η_t^d have mean zero for each d, and satisfy

$$\begin{aligned} &\varepsilon_t^d \perp\!\!\!\perp (\{d_\tau\}_\tau,\{x_\tau^*\}_\tau,\{w_\tau\}_\tau,\{\varepsilon_\tau\}_{\tau\neq t}) && \text{for all } t \\ &\eta_t^d \perp\!\!\!\perp (d_\tau,x_\tau^*,w_\tau) && \text{for all } \tau < t \text{ for all } t. \end{aligned}$$

where $\varepsilon_t = (\varepsilon_t^0,\varepsilon_t^1,\ldots,\varepsilon_t^{\bar{d}})$,

Assumption A.4′. For each d, $((d_{t-1}=d)>0$ and the following matrix is nonsingular for each of $d'=d$ and $d'=0$.

$$\begin{bmatrix} 1 & \mathrm{E}[w_{t-1}\mid d_{t-1}=d,d_{t-2}=d'] & \mathrm{E}[x_{t-1}\mid d_{t-1}=d,d_{t-2}=d'] \\ \mathrm{E}[w_{t-1}\mid d_{t-1}=d,d_{t-2}=d'] & \mathrm{E}[w_{t-1}^2\mid d_{t-1}=d,d_{t-2}=d'] & \mathrm{E}[x_{t-1}w_{t-1}\mid d_{t-1}=d,d_{t-2}=d'] \\ \mathrm{E}[w_t\mid d_{t-1}=d,d_{t-2}=d'] & \mathrm{E}[w_{t-1}w_t\mid d_{t-1}=d,d_{t-2}=d'] & \mathrm{E}[x_{t-1}w_t\mid d_{t-1}=d,d_{t-2}=d'] \end{bmatrix}$$

https://doi.org/10.1017/S0266466617000081 Published online by Cambridge University Press

Assumption A.5′. The random variables w_t and x_t^* have bounded conditional first moments given (d_t, d_{t-1}). The conditional characteristic functions of w_t and x_t^* given (d_t, d_{t-1}) do not vanish on the real line, and are absolutely integrable. The conditional characteristic function of (x_{t-1}^*, w_t) given $(d_{t-1}, d_{t-2}, w_{t-1})$ and the conditional characteristic function of x_t^* given (w_t, d_{t-1}) are absolutely integrable. Random variables ε_t and η_t^d have bounded first moments and absolutely integrable characteristic functions that do not vanish on the real line.

Similar discussions to those made of Assumptions A.2, A.3, A.4, and A.5 are relevant for Assumptions A.2′, A.3′, A.4′, and A.5′. Because x_t^* is a unit-less unobserved variable, there would be a continuum of observationally equivalent set of $(\delta^0, \ldots, \delta^{\bar{d}})$ and distributions of $(\varepsilon_t^0, \ldots, \varepsilon_t^{\bar{d}})$, unless we normalize δ^d for one of the choices d. We, therefore, make the following assumption in addition to the baseline assumptions.

Assumption A.6. With no loss of generality, we normalize $\delta^0 = 1$.

Under this set of assumptions, we obtain the following closed-form identification result.

THEOREM A.2 (Closed-form identification). *If Assumptions A.1, A.2′, A.3′, A.4′, A.5′, and A.6 are satisfied, then the four components* $f_1(d_t|w_t, x_t^*)$, $f_2(w_t|d_{t-1}, w_{t-1}, x_{t-1}^*)$, $f_3(x_t^*|d_{t-1}, w_{t-1}, x_{t-1}^*)$, $f_4(x_t|d_{t-1}, x_t^*)$ *of the Markov kernel*

$$f(d_t, w_t, x_t^*, x_t|d_{t-1}, w_{t-1}, x_{t-1}^*, x_{t-1})$$

are identified by closed-form formulas.

Proof is provided in Section B.2 in supplementary material to this article, available at Cambridge Journals Online (journals.cambridge.org/ect).

A.3. Extension 3: Relaxing the Independent Evolution

In the basic model with an unobserved state x_t^* and an observed state w_t presented in Section A.1, the requirement of the conditionally independent evolution of x_t^* and w_t was mentioned to be restrictive depending on an application. In this section, we propose how to relax this assumption of conditionally independent evolution. We first decompose the Markov kernel as follows.

Assumption A.7 (Independence)**.** The Markov kernel can be decomposed as follows.

$$\begin{aligned} &f(d_t, w_t, x_t^*, x_t|d_{t-1}, w_{t-1}, x_{t-1}^*, x_{t-1}) \\ &\quad = f_1(d_t|w_t, x_t^*)\, f_2(w_t, x_t^*|d_{t-1}, w_{t-1}, x_{t-1}^*)\, f_3(x_t|x_t^*), \end{aligned}$$

where the four components represent

$f_1(d_t\|w_t, x_t^*)$	conditional choice probability (CCP);
$f_2(w_t, x_t^*\|d_{t-1}, w_{t-1}, x_{t-1}^*)$	transition rule for the observed state variable;
$f_3(x_t\|x_t^*)$	proxy model.

As with Assumption A.2, this independence assumption is key to our closed-form identification results, and is better understood in the context of the standard independence assumptions used in the dynamic discrete choice literature if we discuss primitive conditions. Consider the following four independence conditions.

https://doi.org/10.1017/S0266466617000081 Published online by Cambridge University Press

(i) Rust's conditional independence assumption:

$$f(d_t, x_t^*, w_t \mid d_{t-1}, x_{t-1}^*, w_{t-1}) = f_1(d_t \mid x_t^*, w_t) \cdot f_2(x_t^*, w_t \mid d_{t-1}, x_{t-1}^*, w_{t-1})$$

(ii) The true state x_t^* is a sufficient statistic for the proxy x_t:

$$f(x_t \mid d_t, x_t^*, w_t, d_{t-1}, x_{t-1}^*, w_{t-1}, x_{t-1}) = f_3(x_t \mid x_t^*)$$

(iii) The irrelevance of lagged variables to the choice d_t given the true state (x_t^*, w_t):

$$f(d_t \mid x_t^*, w_t, d_{t-1}, x_{t-1}^*, w_{t-1}, x_{t-1}) = f_1(d_t \mid x_t^*, w_t)$$

(iv) The irrelevance of the proxy x_t to the state transition:

$$f(x_t^*, w_t \mid d_{t-1}, x_{t-1}^*, w_{t-1}, x_{t-1}) = f_2(x_t^*, w_t \mid d_{t-1}, x_{t-1}^*, w_{t-1}).$$

Parts (i)–(iv) are together equivalent to Assumption A.2. As in Section A.1, parts (i), (iii), and (iv) are less objectionable in the spirit of the dynamic discrete choice literature. Part (ii) is the only new condition that we are invoking in Assumption A.2 compared to the existing literature. Part (ii) is satisfied if the measurement error, as defined as the difference between proxy x_t and true state x_t^*, is independent of $(d_t, x_t^*, w_t, d_{t-1}, x_{t-1}^*, w_{t-1}, x_{t-1})$. Note that part (v), which we required in Section A.1, is now absent from the current section. This elimination of the fifth part is allowed under the following condition replacing Assumption A.3 of Section A.1.

Assumption A.8 (Semi-parametric restrictions on the unobservables)**.** The transition rule for the unobserved state variable and the state-proxy relation are semiparametrically specified by

$$f_2\left(w_t, x_t^* | d_{t-1}, w_{t-1}, x_{t-1}^*\right): \qquad x_t^* = \alpha^d + \beta^d w_{t-1} + \gamma^d x_{t-1}^* + \eta_t^d \quad \text{if } d_{t-1} = d \tag{A.3}$$

$$f_3\left(x_t | x_t^*\right): \qquad x_t = x_t^* + \varepsilon_t, \tag{A.4}$$

where ε_t and η_t^d have mean zero for each d, and satisfy

$$\begin{aligned} &\varepsilon_t \perp\!\!\!\perp (\{d_\tau\}_\tau, \{x_\tau^*\}_\tau, \{w_\tau\}_\tau, \{\varepsilon_\tau\}_{\tau \neq t}) && \text{for all } t \\ &\eta_t^d \perp\!\!\!\perp \left(d_\tau, x_\tau^*, w_\tau\right) \mid w_t && \text{for all } \tau < t \text{ for all } t. \end{aligned}$$

The only difference from Assumption A.3 is the last conditional independence condition $\eta_t^d \perp\!\!\!\perp \left(d_\tau, x_\tau^*, w_\tau\right) \mid w_t$, which has replaced the unconditional independence condition $\eta_t^d \perp\!\!\!\perp \left(d_\tau, x_\tau^*, w_\tau\right)$. In this new framework, the idiosyncratic error η_t^d for the law of unobserved state transition can depend on w_t, effectively allowing for the conditionally dependent evolution of x_t^* and w_t. However, conditionally on w_t, the idiosyncratic error η_t^d has to be independent of the past. We use the following version of the rank condition in the current section.

Assumption A.9 (Testable rank condition)**.** $\Pr(d_{t-1} = d) > 0$ and the following matrix is nonsingular for each d.

$$\begin{bmatrix} 1 & \mathrm{E}[w_{t-1} \mid d_{t-1} = d] & \mathrm{E}[x_{t-1} \mid d_{t-1} = d] \\ \mathrm{E}[d_{t-2} \mid d_{t-1} = d] & \mathrm{E}[w_{t-1} d_{t-2} \mid d_{t-1} = d] & \mathrm{E}[x_{t-1} d_{t-2} \mid d_{t-1} = d] \\ \mathrm{E}[d_{t-3} \mid d_{t-1} = d] & \mathrm{E}[w_{t-1} d_{t-3} \mid d_{t-1} = d] & \mathrm{E}[x_{t-1} d_{t-3} \mid d_{t-1} = d] \end{bmatrix}$$

THEOREM A.3. *If Assumptions A.1, A.5, A.7, A.8, and A.9 are satisfied, then the four components* $f_1\left(d_t|w_t, x_t^*\right)$, $f_2\left(w_t, x_t^*|d_{t-1}, w_{t-1}, x_{t-1}^*\right)$, *and* $f_3\left(x_t|x_t^*\right)$ *of the Markov kernel*

$$f\left(d_t, w_t, x_t^*, x_t|d_{t-1}, w_{t-1}, x_{t-1}^*, x_{t-1}\right)$$

are identified with closed-form formulas.

Proof is provided in Section B.3 in supplementary material to this article, available at Cambridge Journals Online (journals.cambridge.org/ect).

A.4. Extension 4: Bivariate Unobserved State Variables

In the baseline model and the above extensions, we consider models with a univariate unobserved state variable. In the current section, we consider an alternative extension where there are two unobserved state variables, x_t^* and y_t^*. This framework may be relevant to the model of Cunha, Heckman and Schennach (2010), where cognitive and noncongnitive abilities are considered as two unobserved state variables. We replace Assumptions 1–5 in the baseline model by the following five assumptions.

Assumption A.10 (First-order markov process)**.** The triple $\left\{d_t, x_t^*, x_t, y_t^*, y_t\right\}$ jointly follows a first-order Markov process.

Assumption A.11 (Independence)**.** The Markov kernel can be decomposed as

$$f\left(d_t, x_t^*, x_t, y_t^*, y_t|d_{t-1}, x_{t-1}^*, x_{t-1}, y_{t-1}^*, y_{t-1}\right) = f_1\left(d_t|x_t^*, y_t^*\right) f_{2x}\left(x_t^*|d_{t-1}, x_{t-1}^*\right) \times f_{2y}\left(y_t^*|d_{t-1}, y_{t-1}^*\right) f_{3x}\left(x_t|x_t^*\right) f_{3y}\left(y_t|y_t^*\right),$$

where the five components represent

$f_1\left(d_t\|x_t^*, y_t^*\right)$	conditional choice probability (CCP);
$f_{2x}\left(x_t^*\|d_{t-1}, x_{t-1}^*\right)$	transition rule for the unobserved state variable x_t^*;
$f_{2x}\left(y_t^*\|d_{t-1}, y_{t-1}^*\right)$	transition rule for the unobserved state variable y_t^*;
$f_{3x}\left(x_t\|x_t^*\right)$	proxy model for x_t^*; *and*
$f_{3y}\left(y_t\|y_t^*\right)$	proxy model for y_t^*.

Assumption A.12 (Semi-parametric restrictions on the unobservables)**.** The transition rule for the unobserved state variables and the state-proxy relation are semiparametrically specified by

$$\begin{aligned}
f_{2x}\left(x_t^*|d_{t-1}, x_{t-1}^*\right): &\quad x_t^* = \alpha^{x,d} + \gamma^{x,d} x_{t-1}^* + \eta_t^{x,d} &\quad \text{if } d_{t-1} = d\\
f_{2y}\left(y_t^*|d_{t-1}, y_{t-1}^*\right): &\quad x_t^* = \alpha^{y,d} + \gamma^{y,d} y_{t-1}^* + \eta_t^{y,d} &\quad \text{if } d_{t-1} = d\\
f_{3x}\left(x_t|x_t^*\right): &\quad x_t = x_t^* + \varepsilon_t^x &\\
f_{3y}\left(y_t|y_t^*\right): &\quad y_t = y_t^* + \varepsilon_t^y &
\end{aligned}$$

where ε_t^x, ε_t^y, $\eta_t^{x,d}$, and $\eta_t^{y,d}$ have mean zero for each d, and satisfy

$$\begin{aligned}
\varepsilon_t^x &\perp\!\!\!\perp (\{d_\tau\}_\tau, \{x_\tau^*\}_\tau, \{y_\tau^*\}_\tau, \{\varepsilon_\tau^x\}_{\tau\neq t}, \{\varepsilon_\tau^y\}_\tau) &\quad \text{for all } t\\
\varepsilon_t^y &\perp\!\!\!\perp (\{d_\tau\}_\tau, \{x_\tau^*\}_\tau, \{y_\tau^*\}_\tau, \{\varepsilon_\tau^x\}_\tau, \{\varepsilon_\tau^y\}_{\tau\neq t}) &\quad \text{for all } t\\
\eta_t^{x,d} &\perp\!\!\!\perp \left(\{d_\tau\}_{\tau<t}, \{x_\tau^*\}_{\tau<t}\right) &\quad \text{for all } t\\
\eta_t^{y,d} &\perp\!\!\!\perp \left(\{d_\tau\}_{\tau<t}, \{y_\tau^*\}_{\tau<t}\right) &\quad \text{for all } t.
\end{aligned}$$

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Assumption A.13 (Testable rank condition)**.** $\Pr(d_{t-1} = d) > 0$ and the following matrices are nonsingular for each d.

$$\begin{bmatrix} 1 & \mathrm{E}[x_{t-1} \mid d_{t-1} = d] \\ \mathrm{E}[d_{t-2} \mid d_{t-1} = d] & \mathrm{E}[x_{t-1} d_{t-2} \mid d_{t-1} = d] \end{bmatrix}$$
$$\begin{bmatrix} 1 & \mathrm{E}[y_{t-1} \mid d_{t-1} = d] \\ \mathrm{E}[d_{t-2} \mid d_{t-1} = d] & \mathrm{E}[y_{t-1} d_{t-2} \mid d_{t-1} = d] \end{bmatrix}$$

Assumption A.14 (Regularity)**.** The random variables x_t^* and y_t^* have a bounded conditional first moments given d_t. The conditional characteristic function of x_t^* given $d_t = d$ and the conditional characteristic function of y_t^* given $d_t = d$ do not vanish on the real line, and are absolutely integrable. Random variables ε_t^x, ε_t^y, $\eta_t^{x,d}$, and $\eta_t^{y,d}$ have bounded first moments and have absolutely integrable characteristic functions that do not vanish on the real line.

While Assumptions A.11 and A.12 imply that the two unobserved state variables x_t^* and y_t^* are *conditionally* independent, we emphasize that they are not *unconditionally* independent. The discrete choice d_{t-1} in the last time and the correlated past state (x_{t-1}^*, y_{t-1}^*) induces a correlation between x_t^* and y_t^*. Under the five assumptions stated above, we obtain the following closed-form identification result for the five components of the Markov kernel.

THEOREM A.4. *If Assumptions A.10, A.11, A.12, A.13, and A.14 are satisfied, then the five components,* $f_1\left(d_t | x_t^*, y_t^*\right)$, $f_{2x}\left(x_t^* | d_{t-1}, x_{t-1}^*\right)$, $f_{2y}\left(y_t^* | d_{t-1}, y_{t-1}^*\right)$, $f_{3x}\left(x_t | x_t^*\right)$, *and* $f_{3y}\left(y_t | y_t^*\right)$ *are identified with closed-form formulas.*

The proof is provided in Section B.4 in supplementary material to this article, available at Cambridge Journals Online (journals.cambridge.org/ect).

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