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Dynamic decisions under subjective expectations: A structural analysis[☆]

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ABSTRACT

We study dynamic discrete choice models without assuming rational expectations. Agents' beliefs about state transitions are subjective, unknown, and may differ from their objective counterparts. We show that agents' preferences and subjective beliefs are identified in both finite and infinite horizon models. We estimate the model primitives via maximum likelihood estimation and demonstrate the good performance of the estimator by Monte Carlo experiments. Using the Panel Study of Income Dynamics (PSID) data, we illustrate our method in an analysis of women's labor participation. We find that workers do not hold rational expectations about income transitions.

1. Introduction

Decision making under uncertainty—including educational choice, labor participation, and occupational choice—is a prominent theme in economics. In the literature, agent choices are modeled as the optimal solution to an expected utility maximization problem. Expected utility is computed using agent beliefs about choice-specific future outcomes (e.g., a woman's beliefs about household income, conditional on her labor participation decision). That is, observed choices are determined not only by the agents' preferences but also by their beliefs. A central problem in this literature is to infer agent preferences from the choices observed in the data by using the connection among preferences, beliefs, and choices. Thus, information about agent beliefs is crucial for the inference of preferences from observed choices.

Unfortunately, because the econometrician typically does not observe agent beliefs in practice, certain assumptions are usually imposed. One ubiquitous assumption is that agent beliefs are rational, such that agents' subjective beliefs about future uncertainty will coincide with the distribution of *ex-post* realized outcomes. That assumption may be problematic. First, Manski (1993a) points out that observed choices can be consistent with multiple combinations of beliefs and preferences, and Manski (1993b) shows that even a learning process may not justify rational expectations. Moreover, some recent studies, by comparing survey data on agents' subjective beliefs with their objective counterparts (see

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e.g., Heimer et al., 2019 and Cruces et al., 2013, among others), have documented systematic discrepancies between the two. Furthermore, Esponda and Pouzo (2015) theoretically justify that agents can hold biased beliefs in the steady state. Not surprisingly, violation of the rational expectations assumption may induce biased estimation of agent preferences and misleading counterfactual results. One dominating solution to this problem in the literature is to first solicit agent beliefs (see Manski, 2004 for a review) and then to study agent decisions under those beliefs (see e.g., Van der Klaauw, 2012). Nevertheless, it is costly to obtain information on agent beliefs. Moreover, it is impossible to collect agents' subjective beliefs for some historical datasets.

In the existing literature, there is little knowledge of what can be achieved without solicited subjective beliefs or a known link between such beliefs and some observables, for example, the assumption of rational or myopic expectations. In this paper, we provide a first positive result: we can recover both agents' preferences and their subjective beliefs from their observed choices. Specifically, we consider a standard dynamic discrete choice (DDC) model in which agents may have subjective beliefs about a state variable's transition process, though it is unknown to the econometrician. We provide a unified identification strategy for identifying agent preferences and their subjective beliefs, in both finite and infinite horizon models. The identification relies mainly on variations provided by variables affecting agent conditional choice probabilities (CCPs) but being excluded from beliefs and preferences. Our identification results apply to both homogeneous and heterogeneous beliefs.

We show that when agents' subjective beliefs are homogeneous, they can be identified and estimated from the CCPs in both finite-horizon and infinite-horizon models. Based on the insight of under-identification result—e.g., Rust (1994) and Magnac and Thesmar (2002)—we address identification of DDC models assuming that the discount factor and the distribution of agents' unobserved preference shocks are known. Our methodology then identifies agents' subjective beliefs on state transitions as a closed-form solution to a set of conditions that are induced from Bellman equations, using the insight in Hotz and Miller (1993). In finite horizon models, the key identification assumptions are that: subjective beliefs and preferences are time-invariant; and subjective beliefs are partially known to the econometrician. In addition to these two assumptions, identification of infinite horizon models requires further exclusion restrictions.

In the finite horizon framework, we first explore variations of CCPs over time to identify both agents' beliefs and their preferences. If preferences and beliefs are time-invariant, then for a given state CCPs would only change over time because of proximity to the terminal period. Therefore, under the same state the first difference in CCPs must be attributed to the first difference of continuation values, integrated over all possible future states based on agent subjective beliefs. We then control the impact of continuation values on CCPs via their recursive relationship. Thus it is possible to disentangle beliefs from continuation values using multiple, consecutive time periods of data. Essentially identification requires stationarity of preferences and beliefs so that time shifts CCPs without altering preferences or beliefs. Identification also requires that subjective beliefs are known to the econometrician for an action, or for an action given a state. This is because for any action CCP depends on choice-specific values of all actions, relative to a reference action. These relative choice-specific values further rely on the relative subjective beliefs. Consequently, partially known beliefs are necessary for identifying subjective beliefs for all actions and states. In addition, identification requires a sufficiently large number of periods of data—more than the cardinality of the state variable's support. This requirement may limit the applicability of our identification results in some empirical studies where the support of the state variable is large. Fortunately, we prove that the required number of periods for identification is greatly reduced if there is an additional state variable whose transition is known and independent of the transition of the state variable with subjective beliefs. If this additional state variable takes the same or more values than the state variable with subjective transitions, then four consecutive periods of data are sufficient for identification.

In infinite horizon models, the stationarity of CCPs rules out the possibility of using time variations for identification. Instead, we introduce an additional state variable to achieve identification. This variable evolves independently from the variable with subjective transitions; it has a known transition process; and it enters the flow utility in a particular way. The identification argument here is similar to that of the finite horizon model. Specifically, the CCPs for different values of this additional state variable reveal information about agent flow utility, beliefs, and the *ex-ante* value function. The identification then relies on the following conditions: (1) We impose exclusion restrictions on utility; (2) We assume that both preferences and subjective beliefs are known for a reference action; (3) The additional variable takes at least the same number of possible values as the state variable on which agents hold subjective beliefs.

Our identification strategies also apply to DDC models where agents hold heterogeneous subjective beliefs and/or preferences. Assuming that agents are classified into finite unobserved types, with agents of the same type holding homogeneous beliefs and preferences, we first prove that type-specific CCPs can be identified nonparametrically. This step uses methodologies in measurement error, e.g., Hu (2008). Once the type-specific CCPs are recovered, one can apply the identification results developed for homogeneous beliefs to identify the type-specific subjective beliefs and/or preferences.

We propose a maximum likelihood estimator to estimate agents' preferences and their subjective beliefs in both finite and infinite horizon cases. Our Monte Carlo experiments show that the proposed estimator performs well with moderate sample sizes. That performance is maintained when the data are generated under rational expectations. Furthermore, we find that imposing rational expectations leads to inconsistent estimation of payoff primitives if the data are generated from subjective beliefs that differ from their objective counterparts.

We illustrate our methodology by analyzing women's labor participation using Panel Study of Income Dynamics (PSID) data. To decide whether the wife should join the labor force or stay at home, the household needs to perceive how the wife's labor force status would affect future household income. We divide household income into three groups: low, medium, and high. Our estimation results reveal clear discrepancies between agents' subjective beliefs about state transitions and the objective reality. Households with a non-working wife are overly pessimistic about their income transitions; those with a working wife are less so. We also show that agents have "asymmetric" beliefs about income transitions. Conditional on not working, agents with medium income believe that their income is very likely to remain at medium. However, agents with high income are more pessimistic. They believe that income is almost certain to drop to medium, while the objective probability that income drop is only 0.24. We test the hypothesis that agents hold rational expectations and decisively reject the null hypothesis. We further simulate agent CCPs under both subjective beliefs and rational expectations. Our results suggest that having subjective beliefs increases the probability of working as compared to rational expectations. The effects are heterogeneous across income levels: women with low income are more likely to work under their subjective beliefs than those with medium or high income.

This paper is related to the rapidly growing literature on subjective beliefs. Relaxing rational expectations in DDC models, or in decision models in general, is of both theoretical and empirical importance. Manski (2004) advocates using data on subjective beliefs in empirical decision models. In related literature, a substantial effort has been invested in collecting data on agents' subjective beliefs, so that the econometrician can directly use those beliefs to study agents' behaviors under uncertainty. For example, Van der Klaauw and Wolpin (2008) study retirement and savings using a DDC model in which information about agents' subjective beliefs about their own retirement age, anticipated longevity, and future changes in the Social Security program comes from surveys. Zafar (2011, 2013) studies schooling choice using survey data on students' subjective beliefs. Wang (2014) uses individuals' subjective longevity beliefs to explain adults' smoking decisions. d'Haultfoeuille et al. (2018) provide a new test of rational expectations based on the marginal distributions of realizations and solicited subjective beliefs. Acknowledging the scarcity of data on beliefs, we take a distinct approach from this literature: we focus on inferring agents' subjective beliefs from their choices.

This paper also contributes to a growing literature on the identification of DDC models. Rust (1994) provides some non-identification results for the infinite-horizon case. Magnac and Thesmar (2002) further determine the exact degree of under-identification and explore the identifying power of some exclusion restrictions. Kasahara and Shimotsu (2009) and Hu and Shum (2012) consider identification of DDC models with unobserved heterogeneity/state variables. Abbring and Daljord (2020) identify the discount factor using an exclusion restriction on agent preferences. Abbring (2010) provides an excellent review on identification of DDC models. In all of these papers, the assumption of rational expectations is imposed, and it plays an important role for identification. We relax the assumption of rational expectations and propose an original argument to identify agents' subjective beliefs. The estimated subjective beliefs can be used to test the widely imposed assumption of rational expectations.

Our paper is also related to Aguirregabiria and Magesan (2020), which study identification and estimation of dynamic games where the strategic interaction among players is crucial. In contrast to the existing literature that assumes Nash equilibrium, Aguirregabiria and Magesan (2020) identify players' payoffs and their beliefs about rivals' behaviors while allowing those beliefs to be inconsistent with their equilibrium counterparts. Their paper relies on exclusion restrictions and partially unbiased beliefs for identification, which is similar to that in our paper. However, their paper differs from ours in that players still have rational expectations about state transitions in their model. It is unclear how their identification strategy would apply to our setting. Moreover, in finite horizon framework, we mainly exploit variations of CCPs over time with/without an additional state variable for identification, while Aguirregabiria and Magesan (2020) only explore the identification power of state variables.

The remainder of this paper is organized as follows. Section 2 presents DDC models with subjective beliefs. Section 3 proposes identification results for the finite horizon case. Section 4 describes identification strategies that rely on exclusion restrictions. Section 5 extends identification results to a model with heterogeneous beliefs and/or preferences. Section 6 discusses estimation and provides Monte Carlo evidence. Section 7 focuses on women's labor participation, applying our method to PSID data. Section 8 concludes. Proofs are presented in Appendix.

2. DDC models with subjective beliefs

In this section, we describe a DDC model where agents' rational expectations are relaxed and then present some basic assumptions.

In each discrete time period $t = 1, 2, \ldots, T$ (T can be finite or infinite), a single agent chooses an action a_t from a finite set of actions, $\mathcal{A} = \{1, \ldots, K\}, K \geq 2$, to maximize her expected lifetime utility. The utility-relevant state variables in period t consist of two parts, x_t and ϵ_t, where x_t is the state variable observed by the econometrician, and ϵ_t is a vector of unobserved choice-specific shocks, i.e., $\epsilon_t = (\epsilon_t(1), \ldots, \epsilon_t(K))$. We assume that the observed state variable x_t is discrete and takes values in $\mathcal{X} \equiv \{1, \ldots, J\}, J \geq 2$. Both state variables are known to the agent at the beginning of period t. The agent then makes choice a_t and obtains per-period utility $u(x_t, a_t, \epsilon_t)$. There is uncertainty regarding future states, which is governed by an exogenous mechanism and is assumed to be a Markov process, given the agent's choice. Specifically, given the current state (x, ϵ) and agent choice a, the state variables in the next period (x', ϵ') are determined by the transition function $f(x', \epsilon'|x, \epsilon, a)$. We suppress period subscripts and use prime to represent the next period for ease of notation. Following the existing literature, we impose the following assumption on the transition function.

Assumption 1 (*Conditional Independence*). (a) The observed and unobserved state variables evolve independently conditional on x and a. That is,

$$f(x', \epsilon'|x, a, \epsilon) = f(x'|x, a)f(\epsilon'|\epsilon, a).$$

(b) The unobserved state variables over time periods and actions are independent and identically distributed (i.i.d.) draws from the mean zero type-I extreme value distribution.[1]

Under Assumption 1, the transition process can be simplified as

$$f(x', \epsilon'|x, a, \epsilon) = f(x'|x, a)f(\epsilon').$$

Because the agent is forward-looking and her choice involves intertemporal optimization, her beliefs about the state transition play an essential role in her decision-making process. Let $s(x'|x, a)$ be the agent's subjective beliefs about the transition of the observed state. In the literature of DDC models, a ubiquitous assumption is that agents have perfect beliefs or rational expectations—i.e., $s(x'|x, a) = f(x'|x, a)$ for all x', x and a—and this is crucial for identifying and estimating DDC models (e.g., see Magnac and Thesmar, 2002). Unfortunately, this assumption is very restrictive. Manski (1993b) points out that even a learning process may not justify rational expectations if the law of motion changes between the two cohorts due to some macro-level shocks, or if the earlier cohort's history cannot be fully observed. Moreover, the recent literature documents violations of rational expectations when comparing survey data on agents' subjective beliefs with their objective counterparts and examining the impact of such violations on agent choices. For example, Heimer et al. (2019) show that surveyed beliefs about mortality over the life cycle differ substantially from actuarial statistics from the Social Security Administration. This discrepancy leads the young to under-save by 30% and causes retirees to draw down on their assets 15% more slowly than would be ideal. Cruces et al. (2013) provide evidence of agents' biased perception of the income distribution. And, in the literature on structural models of oligopoly competition, including Aguirregabiria and Jeon (2020), firms also may have biased beliefs about model primitives.

Motivated by the theoretical argument above and the empirical evidence, we relax the assumption of rational expectations in our model. In what follows, we describe the agent's problem in a general framework allowing for subjective beliefs. Then we lay out some basic assumptions and characterize the agent's optimal decision.

In each discrete period, the agent's problem is to decide what action maximizes her expected life-time utility, based on her subjective beliefs about the future evolution of the state variable. The optimization problem is characterized as

$$\max_{a_t \in \mathcal{A}} \sum_{\tau=t,t+1,\ldots} \beta^{\tau-t} E\left[u(x_\tau, a_\tau, \epsilon_\tau)|x_t, a_t, \epsilon_t\right],$$

where $\beta \in [0, 1)$ is the discount factor, $u(x_\tau, a_\tau, \epsilon_\tau)$ is the flow utility, and the expectation is taken over all future actions and states, based on the agent's subjective beliefs $s(x'|x, a)$. These beliefs are a complete set of conditional probabilities that satisfy the following properties:

Assumption 2 (*Valid Stationary Belief*). Agents' subjective beliefs about the transition probabilities of the observed state variable satisfy the following conditions:

(a) $\sum_{x' \in \mathcal{X}} s(x'|x, a) = 1$ and $s(x'|x, a) \geq 0$ for any $x \in \mathcal{X}$ and $a \in \mathcal{A}$.
(b) $s(x'|x, a)$ is time-invariant.

Assumption 2(a) presents some minimum requirements for subjective beliefs as probabilities. Part (b) restricts subjective beliefs to be stationary and rules out the possibility that agents update their beliefs about the transition through learning. There are several reasons why subjective beliefs could be stationary and could differ from their objective counterparts. First, suppose that agents update their beliefs based on some historical information. They may not be able to learn the true objective state transitions even in the steady state. For example, Esponda and Pouzo (2015) consider a dynamic framework where agents have a prior over a set of possible transitions. They show that if this set does not contain the true transition, then agents can hold biased beliefs about the transition even in the steady state. Second, it is possible that agents only update their beliefs after accumulating sufficient evidence, rather than in every period. For example, Coutts (2019) finds in experiment that only 9% subjects update their beliefs in every round. In such a case, it would be reasonable to assume that beliefs would remain the same for some periods.

Assumption 2(b) is an approximation of agent beliefs about state transitions. It might be strong and unrealistic in some applications, where learning should be incorporated. However, modeling learning can be very challenging. Empirically, it is unclear what information agents incorporate into their learning process; e.g., agents may learn from their own experience, their cohort's (Manski, 1993b), or both. An incomplete understanding of the sources of the underlying learning process complicates the framework in either theoretical or empirical analyses. Thus, modeling agents' learning about their subjective beliefs is still an open question so we leave it to future work.

[1] The assumption of type-I extreme value distribution is for ease of illustration. As long as the distribution is known and absolutely continuous, our identification argument holds.

It is worth noting that assuming stationary subjective beliefs is less restrictive than assuming stationary rational expectations. Our model with subjective beliefs nests the existing DDC model with stationary rational expectations as a special case. No matter whether agents have subjective beliefs or rational expectations, the transitions of the state variable are still governed by the objective distribution $f(x'|x, a)$, while agents' optimal behaviors depend on their perception of the objective transition.

Next, we present an assumption about agent preferences following the existing literature (e.g., Rust, 1987).

Assumption 3 (*Stationary and Additively Separable Preference*). The flow utility is time-invariant.[2] The unobserved state is assumed to enter the preference additively and separably, i.e., $u(x, a, \epsilon) = u(x, a) + \epsilon(a) \equiv u_a(x) + \epsilon(a)$ for any $a \in \mathcal{A}$.

The stationarity and additive separability of agent utility imposed in Assumption 3 are used widely in the literature. Consequently, we can represent the agent's optimal choice a_t, which depends on the state, x_t and ϵ_t, in period t as

$$a_t = \arg\max_{a\in\mathcal{A}} \Big\{ u_a(x_t) + \epsilon(a) + \beta \sum_{x'\in\mathcal{X}} V_{t+1}(x')s(x'|x, a) \Big\},$$

where $V_{t+1}(x)$ is the *ex-ante* or continuation value function in period $t+1$ defined below.

$$V_t(x) = E\left[\max_{a\in\mathcal{A}} \Big\{ u_a(x) + \epsilon(a) + \beta \sum_{x'\in\mathcal{X}} V_{t+1}(x')s(x'|x, a) \Big\} \right], \tag{1}$$

where the expectation is taken with respect to the distribution of ϵ.

In the finite horizon setting, an agent can solve the model using backward induction, starting from the terminal period; this requires the continuation value at the terminal period is known to the agent. In the existing literature, the continuation value at the terminal period could be zero or nonzero, depending on the empirical context. In the infinite horizon setting, stationarity implies that the value function is a fixed point of a contraction mapping (see e.g., Aguirregabiria and Mira, 2010 for details).

Following the existing literature, we characterize agents' optimal behavior using a whole set of probabilities (CCPs) that each action $i \in \mathcal{A}$ is chosen conditional on the observed state in period t, denoted as $p_{t,i}(x)$. Under Assumption 1(b), the agent's optimal behavior can be characterized as

$$p_{t,i}(x) = \frac{\exp(v_{t,i}(x))}{\sum_{a\in\mathcal{A}} \exp(v_{t,a}(x))}, \tag{2}$$

where $v_{t,a}(x)$ is the choice-specific value function for action a conditional on state x,

$$v_{t,a}(x) \equiv u_a(x) + \beta \sum_{x'\in\mathcal{X}} V_{t+1}(x')s(x'|x, a).$$

3. Identification with time variations in CCPs

In this section, we provide sufficient conditions under which agents' subjective beliefs are uniquely determined by their CCPs in a finite horizon framework. The main idea of identification is to build a relationship between observed CCPs and unknown subjective beliefs by exploring the variation of CCPs over time. Specifically, time is excluded from flow utility and beliefs, but it does affect CCPs given its proximity to the terminal period.

Suppose we observe data $\{a_t, x_t\}$, where $t = 1, \ldots, T$, and T is not necessarily the terminal period of agent decisions. Eq. (2) allows us to obtain the log ratio of CCPs for action $i \in \mathcal{A}$ over K $(i \neq K)$ at time t for state x:

$$\begin{aligned} \xi_{t,i,K}(x) &\equiv \log\Big(\frac{p_{t,i}(x)}{p_{t,K}(x)}\Big) = v_{t,i}(x) - v_{t,K}(x) \\ &= \big[u_i(x) - u_K(x)\big] + \beta \sum_{x'\in\mathcal{X}} V_{t+1}(x')\big[s(x'|x, i) - s(x'|x, K)\big]. \end{aligned} \tag{3}$$

The log ratio of CCPs $\xi_{t,i,K}$ captures the likelihood of an agent choosing action i relative to action K in period t in state x. It can be represented in a matrix form as follows.

$$\xi_{t,i,K}(x) \equiv \log\left(\frac{p_{t,i}(x)}{p_{t,K}(x)}\right) = u_i(x) - u_K(x) + \beta\big[S_i(x) - S_K(x)\big]\boldsymbol{V}_{t+1}, t = 1, 2, \ldots, T, \tag{4}$$

where $S_a(x) \equiv [s(x' = 1|x, a), \ldots, s(x' = J-1|x, a)], \forall a \in \mathcal{A}$, is a $1 \times (J-1)$ vector capturing beliefs associated with action a and state x but excluding the element $s(x' = J|x, a)$ because they sum to one; the *ex-ante* value function vector $\boldsymbol{V}_{t+1}$, thus, is constructed as a $(J-1) \times 1$ vector that consists of relative values using J as a reference state, i.e., $\boldsymbol{V}_{t+1} \equiv [V_{t+1}(x=1) - V_{t+1}(J), \ldots, V_{t+1}(x = J-1) - V_{t+1}(J)]'$.

[2] With this assumption, time is excluded from the utility function so that exclusion restrictions hold trivially, as in Bajari et al. (2016).

From Eq. (4), the log ratios of CCP $\xi_{t,i,K}(x)$ are determined by three components: the difference in utilities between action i and K, $u_i(x) - u_K(x)$; the relative continuation value $\boldsymbol{V}_{t+1}$ discounted by β; and the subjective beliefs conditional on action i relative to action K, $S_i(x) - S_K(x)$. We assume that the discount factor β is known because it is not the focus of our paper.[3] To separately recover those components from the observed CCPs, we rely on variables that shift the log ratios of CCPs without affecting the relative flow utility. Time can be such a variable because it is excluded from preferences and beliefs. As a result, we can control the impact of the flow utility on the CCPs by taking the first difference of the log ratio of CCPs for any given state. That is,

$$\Delta\xi_{t,i,K}(x) \equiv \xi_{t,i,K}(x) - \xi_{t-1,i,K}(x) = \beta\big[S_i(x) - S_K(x)\big]\Delta\boldsymbol{V}_{t+1}, t = 2, 3, \ldots, T-1, \tag{5}$$

where $\Delta\boldsymbol{V}_{t+1} \equiv \boldsymbol{V}_{t+1} - \boldsymbol{V}_t$ captures the first difference of the relative *ex-ante* value functions. To proceed, we stack Eq. (5) and collect all $J-1$ equations for $x = 1, 2, \ldots, J-1$ to obtain the following matrix representation of the equation above,

$$\boldsymbol{\Delta\xi}_{t,i,K} = \beta\big[\boldsymbol{S}_i - \boldsymbol{S}_K\big]\boldsymbol{\Delta V}_{t+1}, \tag{6}$$

where $\boldsymbol{\Delta\xi}_{t,i,K} \equiv [\Delta\xi_{t,i,K}(1), \ldots, \Delta\xi_{t,i,K}(J-1)]'$ collects the first difference of the log ratio of CCPs for $J-1$ values of x, and $\boldsymbol{S}_a \equiv [S_a(x=1), \ldots, S_a(x=J-1)]'$ is a $(J-1)\times(J-1)$ matrix that stacks the beliefs associated with action a for $x \in \{1, \ldots, J-1\}$.

Eq. (6) summarizes restrictions implied by the model on subjective beliefs for actions i and K. These restrictions are insufficient for us to identify $\boldsymbol{S}_i$ and $\boldsymbol{S}_K$ because $\boldsymbol{\Delta V}_{t+1}$ is unknown. However, we can control the impact of continuation values on CCPs via its recursive relationship. Because in Eq. (6) the variation in the log ratios of CCPs only reveals information about agent beliefs once we control for the *ex-ante* value function, it is possible for us to disentangle subjective beliefs from continuation values. For this purpose, we obtain extra restrictions for the model primitives by exploring the recursive relation between continuation value functions by backward induction. That is,

$$\boldsymbol{V}_t = -\log \boldsymbol{p}_{t,K} + \boldsymbol{u}_K + \widetilde{\boldsymbol{S}}_K \boldsymbol{V}_{t+1}, \tag{7}$$

where $\widetilde{\boldsymbol{S}}_K$ is defined as a $(J-1)\times(J-1)$ matrix with its jth row $(j = 1, 2, \ldots, J-1)$ being the agent's belief about state transition under action $a = K$ when the current state is $x = j$ relative to that of state $x = J$, i.e., $S_K(j) - S_K(J)$. The vector of log CCPs, $\log \boldsymbol{p}_{t,K}$, and flow utility, $\boldsymbol{u}_K$, are defined analogously. The construction of $\widetilde{\boldsymbol{S}}_K$ is necessary because the vector of the *ex-ante* value function is formulated as the relative *ex-ante* value of state x with respect to the reference state J, i.e., $V_t(x) - V_t(J)$.

Intuitively, Eq. (7) indicates that the *ex-ante* value from optimal behaviors can be expressed as the overall value of choosing action K, $\boldsymbol{u}_K + \widetilde{\boldsymbol{S}}_K\boldsymbol{V}_{t+1}$, and a non-negative adjustment term, $-\log \boldsymbol{p}_{t,K}$, which adjusts for the fact that K may not be the optimal choice. This adjustment term goes to zero as the probability of selecting K goes to one. Note that the recursive relation in Eq. (7) holds for all the choices and is derived from agent's optimization condition via backward induction. Therefore, Eq. (7) provides some restrictions to model primitives besides Eq. (5) but it requires no additional assumptions.

We take advantage of the time-invariant utility again and obtain a recursive relationship for the first difference of the *ex-ante* value function $\boldsymbol{V}_t$,

$$\Delta\boldsymbol{V}_t = -\Delta \log \boldsymbol{p}_{t,K} + \beta\widetilde{\boldsymbol{S}}_K\boldsymbol{\Delta V}_{t+1}. \tag{8}$$

Eqs. (6) and (8) summarize all the restrictions implied by the model in any three consecutive periods. Combining the two equations allows us to disentangle $\boldsymbol{S}_i - \boldsymbol{S}_K$ and $\boldsymbol{\Delta V}_{t+1}$. Specifically, in the first step, we separate beliefs $\boldsymbol{S}_i - \boldsymbol{S}_K$ from *ex-ante* values $\boldsymbol{\Delta V}_{t+1}$ using Eq. (6), which enables us to represent the *ex-ante* value as a function of the beliefs. For this purpose, we make the following assumption.

Assumption 4. There exists one action i, $i \neq K$ such that the $(J-1)\times(J-1)$ belief matrix $\boldsymbol{S}_i - \boldsymbol{S}_K$ is full rank.

To better understand the full rank condition, we explore the restrictions imposed on the model by this assumption. If $J = 2$, i.e., $x \in \{1, 2\}$, the full rank condition is simplified as $s(x' = 1|x = 1, i) \neq s(x' = 1|x = 1, K)$. This implies that the agent believes that her different actions would affect the state transition differently. In case of $J \geq 3$, the full rank condition restricts the $J-1$ belief differences $S_i(x) - S_K(x)$, $x = 1, \ldots, J-1$, to be linearly independent. This full rank condition guarantees that given a set of beliefs, there exists a unique set of *ex-ante* value functions $\boldsymbol{\Delta V}_{t+1}$ that is consistent with the data based on Eq. (6). That $\boldsymbol{S}_i - \boldsymbol{S}_K$ has full rank is sufficient but not necessary for identification.

Under Assumption 4, the first difference of *ex-ante* value functions $\boldsymbol{\Delta V}_{t+1}$ can be expressed explicitly in a closed form,

$$\boldsymbol{\Delta V}_{t+1} = \beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\boldsymbol{\Delta\xi}_{t,i,K}, \quad t = 2, \ldots, T. \tag{9}$$

The recursive relationship in Eq. (8), together with the closed-form expression above, allows us to obtain a moment condition with subjective beliefs being the only unknowns. That is,

$$\beta^{-1}\big[\boldsymbol{S}_i - \boldsymbol{S}_K\big]^{-1}\boldsymbol{\Delta\xi}_{t-1,i,K} = -\Delta\log\boldsymbol{p}_{t,K} + \widetilde{\boldsymbol{S}}_K\big[\boldsymbol{S}_i - \boldsymbol{S}_K\big]^{-1}\boldsymbol{\Delta\xi}_{t,i,K}, \quad t = 3, \ldots, T. \tag{10}$$

[3] We refer to Magnac and Thesmar (2002) and Abbring and Daljord (2020) for the identification of the discount factor β.

Stacking all the conditions for time $t = 3, \dots, T$ leads to the following equation:

$$\left[\widetilde{\boldsymbol{S}}_K[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}, \quad -\beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\right]\Delta\boldsymbol{\xi}_{i,K} = \Delta \log \boldsymbol{p}_K, \tag{11}$$

where $\Delta \log \boldsymbol{p}_K \equiv \left[\Delta \log \boldsymbol{p}_{2,K}, \Delta \log \boldsymbol{p}_{3,K}, \dots, \Delta \log \boldsymbol{p}_{T-1,K}\right]$, with dimensions being $(J-1)$ by $(T-2)$, captures all first differences of log CCPs over time, and $\Delta\boldsymbol{\xi}_{i,K} \equiv \begin{bmatrix} \Delta\boldsymbol{\xi}^1_{i,K} \\ \Delta\boldsymbol{\xi}^2_{i,K} \end{bmatrix} \equiv \begin{bmatrix} \Delta\boldsymbol{\xi}_{3,i,K} & \Delta\boldsymbol{\xi}_{4,i,K} & \cdots & \Delta\boldsymbol{\xi}_{T,i,K} \\ \Delta\boldsymbol{\xi}_{2,i,K} & \Delta\boldsymbol{\xi}_{3,i,K} & \cdots & \Delta\boldsymbol{\xi}_{T-1,i,K} \end{bmatrix}$, with dimensions being $(2J-2)$ by $(T-2)$, collects all the first differences of log ratios of CCPs. Note that the dimensions of $\widetilde{\boldsymbol{S}}_K$ and $\boldsymbol{S}_i - \boldsymbol{S}_K$ are both $(J-1)$ by $(J-1)$. The conditions (11) provide a direct link between the first differences of the log ratios of CCPs and the subjective beliefs. Identification of beliefs relies on the variation of the first differences of log ratios of CCPs over time, which is summarized in matrix $\Delta\boldsymbol{\xi}_{i,K}$. To recover the unknown subjective beliefs from Eq. (11), sufficient variation in the first differences of log ratios of CCPs is required. Such requirements are satisfied under the following assumption.

Assumption 5A. (a) The number of periods is sufficiently large, $T \geq 2J$. (b) The matrix $\Delta\boldsymbol{\xi}_{i,K}$ is of full row rank.

This assumption is imposed on the observed CCPs and thus is empirically testable. Eq. (11) is a linear system with each row of the two matrices $\widetilde{\boldsymbol{S}}_K[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}$ and $\boldsymbol{S}_i - \boldsymbol{S}_K$ being the unknowns. Note that each row of the unknown matrix has $J-1$ parameters while each cell of three consecutive periods of data (e.g., $\Delta\boldsymbol{\xi}_{t,i,K}$, $\Delta\boldsymbol{\xi}_{t-1,i,K}$, and $\Delta \log \boldsymbol{p}_{t,K}$ involve periods $t-1$, t, and $t+1$) provides *one* restriction to the unknown row parameters. Solving this linear system requires (1) the number of restrictions $(T-2)$ is no less than the number of parameters $2(J-1)$, which implies $T \geq 2J$; and (2) the matrix of data $\Delta\boldsymbol{\xi}_{i,K}$ is of full row rank.[4]

Under Assumption 5A, we can get a closed-form solution for the belief matrices $\widetilde{\boldsymbol{S}}_K$ and $\boldsymbol{S}_i - \boldsymbol{S}_K$ from Eq. (11). Note that belief matrix $\widetilde{\boldsymbol{S}}_K$ describes subjective beliefs for choice K in a state $x \in \{1, 2, \dots, J-1\}$ relative to state J. Without further restrictions, we are unable to *fully* identify subjective beliefs conditional on choice K from the identified matrix $\widetilde{\boldsymbol{S}}_K$. To achieve identification, we impose the following assumption to pin down the belief vector for the reference state J, i.e., $\boldsymbol{S}_K(J)$.

Assumption 6A. There exists a state $x \in \{1, 2, \dots, J\}$ under which agents' subjective beliefs about the state transition are known for action K.

The restriction of known subjective beliefs imposed in Assumption 6A is only required to hold for a certain state and action. Without loss of generality, we assume $S_K(J)$ is known. A sufficient condition for the known $S_K(J)$ is that agents hold rational expectations for action K in state J and the econometrician knows this fact. Subjective beliefs need to be partially known due to the nature of DDC models. Specifically, the CCP for any action does not depend solely on its choice-specific value; rather, it depends on the choice-specific values of all actions relative to a reference action, say, K. These relative choice-specific values further rely on beliefs for i relative to K. This implies that partially known beliefs are necessary to identify subjective beliefs for all actions and states. One possibility for relaxing this assumption is to explore other restrictions to the model, e.g., the relationship between subjective beliefs and objective transitions.

We consider an example where a single woman makes dynamic labor participation decisions (to work or not work). The state variable is her income (high, medium, or low), and she has to form beliefs regarding her household future income given current household income and her working status, i.e., the income transition. The assumption of partially known beliefs in this context is reasonable: women with low income who are not working have a very good understanding of their income in the future because in that scenario their only income source is some social welfare programs. Thus, the subjective beliefs are the same as their objective counterparts, conditional on low income and non-working.

Under Assumption 6A, we can identify the subjective beliefs matrix associated with action K, $\boldsymbol{S}_K$. Consequently, the matrix $\boldsymbol{S}_i$ is identified from $\boldsymbol{S}_i - \boldsymbol{S}_K$. The identification results of the *ex-ante* value difference $\Delta\boldsymbol{V}_t$ and belief matrices $\boldsymbol{S}_i$ and $\boldsymbol{S}_K$ can be used to identify belief matrices for actions $i' \neq i$ and $i' \neq K$ using Eq. (6) of action i', which does not require any further assumptions. We summarize our identification results as follows.

Theorem 1. *Suppose that Assumptions 1–4, 5A, and 6A hold. Then the subjective beliefs $s(x'|x, a)$ for $x, x' \in \{1, 2, \dots, J\}$ and $a \in \{1, 2, \dots, K\}$ are identified as a closed-form function of the CCPs, $p_t(a|x)$, for $t = 1, 2, \dots, T$, $T \geq 2J$.*

The identification results in Theorem 1 require at least $2J$ consecutive periods of observations or $2J-2$ cells of three consecutive periods. This requirement could be restrictive in some empirical applications, especially when the state space is large. Next we present an alternative strategy where $J+1$ periods of data are sufficient for identification.

Assumption 5B. (a) The number of periods observed is not smaller than $J+1$, i.e., $T \geq J+1$. (b) The matrix $(\Delta\boldsymbol{\xi}^1_{i,K})' \otimes (\beta\widetilde{\boldsymbol{S}}_K) - (\Delta\boldsymbol{\xi}^2_{i,K})' \otimes I$ is of full column rank.

[4] Assumption 5A(b) guarantees that when $T = 2J$ or $T > 2J$, $\Delta\boldsymbol{\xi}_{i,K}$ has an inverse or a right inverse, respectively.

Assumption 6B. There exists an action $a = K$ such that agents' subjective beliefs about the state transition conditional on this action are known.

The matrix $(\Delta\boldsymbol{\xi}^1_{i,K})'\otimes(\beta\widetilde{\boldsymbol{S}}_K)-(\Delta\boldsymbol{\xi}^2_{i,K})'\otimes I$ is of size $(T-2)\cdot(J-1)$ by $(J-1)\cdot(J-1)$, where I is a $J-1$ by $J-1$ identity matrix, and $\otimes$ denotes the Kronecker product. Assumption 5B is similar to the full rank condition 5A: both require sufficient variation in CCPs over time, and both are empirically testable. Under Assumption 6B, there are $J-1$ unknown parameters in the linear system (11). Following the discussion of Assumption 5A, we need $T-2 \geq J-1$, i.e., $T \geq J+1$ to solve the linear system if $\boldsymbol{S}_K$ is known. Once $\beta\widetilde{\boldsymbol{S}}_K$ is known, Assumption 5B(b) imposes restrictions on the observed log ratios of CCPs, $\Delta\boldsymbol{\xi}^1_{i,K}$, and $\Delta\boldsymbol{\xi}^2_{i,K}$.

Assumption 6B requires that the belief matrix associated with action K is known; i.e., $\boldsymbol{S}_K$ is known, which is stronger than Assumption 6A. For example, we consider a dynamic investment problem, where an agent chooses whether to invest in the stock market or to save in a savings account. The decision is based on her subjective beliefs about the transition of the state variable, wealth. Accumulated wealth in the future conditional on the current wealth and the action of saving should be very straightforward to predict. That is, agents have rational expectations if they put their money in a savings account. In contrast, because of market volatility accumulated future wealth is more difficult to predict if the agent chooses to invest.

In the following theorem, we state the identification result under Assumptions 5B and 6B. We omit the proof because it is similar to the proof of Theorem 1.

Theorem 2. *Suppose that Assumptions 1–4, 5B, and 6B hold. Then the subjective beliefs $s(x'|x,a)$ for $x, x' \in \{1, 2, \ldots, J\}$ and $a \in \{1, 2, \ldots, K-1\}$ are identified as a closed-form function of the CCPs, $p_t(a|x)$, $t = 1, 2, \ldots, T$, $T \geq J+1$.*

Theorems 1–2 demonstrate that it is indeed difficult to disentangle agent beliefs from preferences by observing only their choices over time. Even if beliefs and preferences are time-invariant, identification still requires a sufficiently large number of periods. Moreover, subjective beliefs cannot be point identified unless beliefs are partially known. Theorems 1–2 illustrate the trade-off between normalization of subjective beliefs and the requirement on the data.[5] In particular, if we have more information regarding subjective beliefs, identification will require fewer time periods of data. This is especially helpful in the empirical applications where the state variable takes a large number of values.

In Theorems 1–2, we neither impose restrictions on the continuation value nor require the observation of data in the terminal period. Naturally, if the data for the terminal period are available, and we are willing to assume that the continuation value at the terminal period is zero, then we can achieve identification of the subjective beliefs using fewer periods of data. That is, we can identify the flow utility using the CCP at the terminal period. With identification of the flow utility, there is no need to control for its impact on the log ratio of CCPs by taking difference. Therefore, the fact that the flow utility is identified separately reduces the required number of periods by one. We leave the details of the identification to Appendix.

Remark 1. Given that the discount factor and the distribution of the unobserved state variable are known, and that subjective beliefs are identified, we can non-parametrically identify the utility function relative to action K, i.e., $u_a(x) - u_K(x)$, following Magnac and Thesmar (2002). Furthermore, the discount factor can be identified if we are willing to impose a stronger normalization condition (Assumption 6B) and rank condition (Assumption 5A). This is because the discount factor affects the log ratio of CCPs in a similar way as the belief difference $S_i - S_K$ does in the key equation (4) and more restrictions on beliefs under a stronger normalization provide identifying power for the discount factor β. Specifically, we first identify $\widetilde{\boldsymbol{S}}_K[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}$ and $\beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}$ separately from Eq. (11). We then can identify $\boldsymbol{S}_i - \boldsymbol{S}_K$ with Assumption 6B that $\widetilde{\boldsymbol{S}}_K$ is known. Consequently, both beliefs $\boldsymbol{S}_i$ and discount factor β are identified.

4. Identification with additional state variables

The identification strategy in Section 3 relies on the stationarity of beliefs and preferences. In this section, we extend the identification argument to the case where an additional state variable evolves independently from the state variables that agents hold subjective beliefs. Specifically, suppose there exists an additional state variable w, which is also discrete, $w \in \{1, 2, \ldots, M\}$. We impose the following assumptions on the transition of w.

Assumption 7. (a) The observed state variables x and w evolve independently, and the transition of w is exogenous, i.e,

$$f(x', w'|x, w, a) = f(x'|x, a)f(w'|w).$$

(b) Agents believe that state variables x and w evolve independently and have rational expectations on the evolution of w.

$$s(x', w'|x, w, a) = s(x'|x, a)s(w'|w) = s(x'|x, a)f(w'|w). \tag{12}$$

[5] Although the subjective beliefs in Theorems 1–2 are identified under the assumption that beliefs are partially known, the results could be important and useful in testing whether the assumption of rational expectations is valid.

In addition to the independence of transitions, Assumption 7(a) restricts the state variable w to be at the "macro level" such that agents' actions have no impact on its transition. This restriction can be relaxed in the infinite horizon framework. Assumption 7(b) imposes two restrictions on agents' subjective beliefs. First, agents correctly predict that the two state variables are evolving independently. Second, agents have rational expectations about the transition of state variable w.[6] Note that the independence of the transitions for state variables x and w in Assumption 7(a) is often assumed in the literature (e.g., see the applications of DDC models reviewed in Aguirregabiria and Mira, 2010). The assumption of the analyst knowing the transition of w can be rationalized by the fact that agents often have better understanding of the transition of some variables than of others. Especially if w is a macro-level variable and its transition does not depend on agent choice, agents' subjective beliefs about its transition could be accurate (see e.g., Manski, 2004).

In Assumption 7, the variable w is excluded from the subjective beliefs $s(x'|x, a)$. This is necessary because our objective is to identify subjective beliefs on x. If w also affects agent subjective beliefs about x, then the number of unknowns increases at the same order as the cardinality of the support of w, and there is no benefit from adding this additional state variable.

4.1. Identification of finite horizon models

In the finite horizon framework, both time and the additional state variable, w, can change CCPs without affecting subjective beliefs under Assumption 7. Therefore, the identification of beliefs is easier than the case where we only rely on time. We first rewrite the log ratio of CCPs in period t with the additional state variable.

$$\begin{aligned}\xi_{t,i,K}(x, w) &\equiv \left[u_i(x, w) + \beta \sum_{x', w'} V_{t+1}(x', w')f(w'|w)s(x'|x, i)\right] \\ &\quad - \left[u_K(x, w) + \beta \sum_{x', w'} V_{t+1}(x', w')f(w'|w)s(x'|x, K)\right]. \\ &\equiv u_i(x, w) - u_K(x, w) + \beta(S_i(x) - S_K(x))\boldsymbol{V}_{t+1}F(w), \end{aligned} \tag{13}$$

where $F(w) \equiv [\Pr(w' = 1|w), \ldots, \Pr(w' = M|w)]'$ is a $M \times 1$ vector of probabilities for future state w' conditional on w, and $\boldsymbol{V}_t$ is the $(J-1) \times M$ *ex-ante* value matrix with the (j, k)-th element being the value of $x = j$ relative to $x = J$ with $w = k$, i.e., $\boldsymbol{V}_t \equiv [V_t(x = j, w = k) - V_t(x = J, w = k)]_{j,k}$, for $j = 1, \ldots, J-1$ and $k = 1, \ldots, M$.

Following an identification argument similar to that in Section 3, we assume that preferences are time-invariant so as to rule out the impact of flow utility on the log ratio of CCPs over time. By slight abuse of notation, we use $\Delta\xi_{t,i,K}(x, w)$ to represent the first difference of the log ratios of CCPs when we have the additional state variable w. That is,

$$\begin{aligned}\Delta\xi_{t,i,K}(x, w) &\equiv \xi_{t,i,K}(x, w) - \xi_{t-1,i,K}(x, w) \\ &\equiv \beta[S_i(x) - S_K(x)]\Delta\boldsymbol{V}_{t+1}F(w),\end{aligned} \tag{14}$$

where $\Delta\boldsymbol{V}_{t+1} \equiv \boldsymbol{V}_{t+1} - \boldsymbol{V}_t$ is the first difference of $\boldsymbol{V}_{t+1}$, so it is a $(J-1) \times M$ matrix. The equation above indicates that the change over time in the log ratio of CCPs conditional on states x and w is determined by the belief difference for state x, the difference of the *ex-ante* value functions, and the transition for w.

We collect the moment conditions in (14) with $J-1$ values of x and all values of w to construct the following matrix equation.

$$\Delta\boldsymbol{\xi}_{t,i,K} \equiv \beta\left[\boldsymbol{S}_i - \boldsymbol{S}_K\right]\Delta\boldsymbol{V}_{t+1}\boldsymbol{F}_w, \tag{15}$$

where the matrix of the first difference of log ratios of CCPs $\Delta\boldsymbol{\xi}_{t,i,K}$ is defined similarly to the vector of the first difference of log ratios of CCPs in Section 3 with the (j, k)-th element being the first difference of log ratio of CCPs for $x = j$ and $w = k$, i.e, $\Delta\boldsymbol{\xi}_{t,i,K} \equiv [\Delta\xi_{t,i,K}(x = j, w = k)]_{j,k}$, where $j = 1, \ldots, J-1$ and $k = 1, \ldots, M$, and matrix $\boldsymbol{F}_w$ captures the overall transition matrix of w, i.e., $\boldsymbol{F}_w \equiv [F(w = 1), \ldots, F(w = M)]$. The equation above is the matrix version of Eq. (14).

Following the identification argument in Section 3, we also explore the recursive relation of the value function over time to provide additional restrictions on beliefs. By imposing the full rank condition on $\boldsymbol{S}_i - \boldsymbol{S}_K$ (Assumption 4), combined with the recursive property of the *ex-ante* value function, we obtain the condition with beliefs being the only unknowns as follows.

$$\left[\widetilde{\boldsymbol{S}}_K[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}, \quad -\beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\right]\Delta\boldsymbol{\xi}_{i,K} = \Delta\log\boldsymbol{p}_K\boldsymbol{F}_w, \tag{16}$$

where $\Delta\log\boldsymbol{p}_K$ collects all first difference CCPs over time and across states and is defined as in the previous section; $\Delta\boldsymbol{\xi}_{i,K}$ collects all first differences of the log ratio of CCPs over time and across states, and it is defined similarly to $\Delta\boldsymbol{\xi}_{i,K}$ below Eq. (11) with adjustment using the transition of state w, i.e., $\Delta\boldsymbol{\xi}_{i,K} \equiv \begin{bmatrix} \Delta\boldsymbol{\xi}_{3,i,K}\boldsymbol{F}_w & \Delta\boldsymbol{\xi}_{4,i,K}\boldsymbol{F}_w & \ldots & \Delta\boldsymbol{\xi}_{T,i,K}\boldsymbol{F}_w \\ \Delta\boldsymbol{\xi}_{2,i,K} & \Delta\boldsymbol{\xi}_{3,i,K} & \ldots & \Delta\boldsymbol{\xi}_{T-1,i,K}\end{bmatrix}$. The matrix of the first differences of the log ratio of CCPs, $\Delta\boldsymbol{\xi}_{i,K}$, is of dimension $(2J-2) \times \left(M \cdot (T-2)\right)$.

[6] We can relax this assumption such that $s(w'|w) \neq f(w'|w)$, but $s(w'|w)$ is known to the econometrician.

Eq. (16), similar to Eq. (11), is the key identification equation where variations of CCPs are from time and different values of w. The following assumption concerns the requirements on time and the state variable w to achieve identification.

Assumption 5C. (a) The number of observed periods, T, and the cardinality of the additional state variable's support, M, satisfy $M(T-2) \geq 2J-2$. (b) The matrix $\Delta\boldsymbol{\xi}_{i,K}$ is of full row rank.

Assumption 5C(a) requires that the total number of time periods T is not less than $2+(2J-2)/M$, which can be much smaller than J if M is large. A comparison of (16) and (11) indicates that the additional state variable augments the number of restrictions from 1 to M for any given state value x, while the number of unknown parameters is still $2(J-1)$. Solving the linear system (16) requires the number of restrictions to be relative large, i.e., $M(T-2) \geq 2J-2$, and the matrix $\Delta\boldsymbol{\xi}_{i,K}$ is of full row rank, which is empirically testable.

Under Assumption 5C, we follow the identification strategy in Section 3 to identify all of the subjective beliefs with some normalization. Let $\lceil y \rceil$ denote the minimum integer greater than or equal to y. The results are summarized in the following theorem.

Theorem 3. *Suppose that Assumptions 1–4, 5C, 6A, and 7 hold. Then the subjective beliefs $s(x'|x,a)$ for $x, x' \in \{1,2,\ldots,J\}$ and $a \in \{1,2,\ldots,K-1\}$ are identified as a closed-form function of the CCPs, $p_t(a|x)$, for $t=1,\ldots,T$, $T \geq \lceil \frac{2J-2}{M} \rceil + 2$.*

In Theorem 3 the required number of periods of data reveals complementarity of the additional state variable and time, in terms of providing variation for identification: the required number of periods of data decreases in the cardinality of the additional state variable's support. Moreover, the number of periods required for identification may not depend on the cardinality of the support of x, J. For example, in the case of $M=J$, four periods of data are sufficient for identification; if $M \geq 2J-2$, three periods are sufficient for identification. The fewer periods of data required for identification in Theorem 3 is of great importance in those empirical applications where the state variable may take a large number of possible values.

4.2. Identification of infinite horizon models

The identification strategy in Section 4.1 cannot be directly extended to the infinite horizon framework because CCPs are time-invariant. Instead, we mainly explore variation of CCPs across values of w. For identification, we need to control the impact of flow utility on CCPs when we change the additional state variable w. This is achieved by imposing exclusion restrictions on utility, which are summarized in the following assumption.

Assumption 8. (a) The utility function for choice $a=K$ is normalized, $u_K(x,w)=0$ for any x and w. (b) There exist at least $J-1$ pairs of values for the additional state variable w, such that $\forall i \neq K$, $\forall x \in \mathcal{X}$,

$$u_i(x, w_1^j) = u_i(x, w_2^j), \quad j = 1, \ldots, J-1, \tag{17}$$

where $w^j \equiv \{w_1^j, w_2^j\}$ is the jth pair of $\{w_1, w_2\}$.

Assumption 8(a), the normalization assumption, is widely imposed in the literature of DDC models and discrete games because we can only identify the relative flow utility (Rust, 1994). The exclusion restrictions in Assumption 8(b) have been exploited in the existing literature (see e.g., Abbring and Daljord, 2020). Note that our identification requires that there exist $J-1$ such pairs, which indicates that the state variable w takes at least J values, i.e., $M \geq J$. Moreover, the $J-1$ pairs can be different across actions.

Because of the stationarity in the infinite horizon framework, we drop the index of time in the notation. The log ratio of CCPs can be rewritten as

$$\xi_{i,K}(x,w) \equiv u_i(x,w) - u_K(x,w) + \beta(S_i(x) - S_K(x))\boldsymbol{V}F(w), \tag{18}$$

where $S_i(x)$, $S_K(x)$, $\boldsymbol{V}$, and $F(w)$ are defined analogously to that in Eq. (13).

Unlike in finite horizon models, we cannot rely on variations in the log ratio of CCPs over time to identify beliefs. Instead, we explore variations in the log ratio of CCPs across the special pairs w^j. Under Assumption 8, $u_i(x,w) - u_K(x,w)$ is invariant for any such pairs; therefore, we have

$$\begin{aligned} \Delta\xi_{i,K}(x,w^j) &\equiv \xi_{i,K}(x,w_1^j) - \xi_{i,K}(x,w_2^j) \\ &= \beta(S_i(x) - S_K(x))\boldsymbol{V}[F(w_1^j) - F(w_2^j)] \\ &\equiv \beta(S_i(x) - S_K(x))\boldsymbol{V}\Delta F(w^j), \end{aligned} \tag{19}$$

where $\Delta F(w^j) \equiv F(w_1^j) - F(w_2^j)$ is of dimension $M \times 1$, captures the difference in transition conditional on the future value of the pair w^j.

Recall that for finite horizon models in Theorem 3, we rely on both a relationship similar to Eq. (19) and the recursive nature of the first difference of the value functions $\boldsymbol{\Delta V}_t$ for identification. However, such a strategy is infeasible for infinite horizon models because there does not exist a similar recursive relationship for the value function $\boldsymbol{V}$ due to its stationarity. Thus, we need to recover the unknown $\boldsymbol{V}$, which depends on both preferences and beliefs, to identify beliefs $S_i(x) - S_K(x)$ from Eq. (19).

We show in the following lemma that $\boldsymbol{V}$ is identified if the subjective beliefs associated with the reference action K are known, i.e., Assumption 6B holds, and the preference associated with this reference action is normalized.

Lemma 1. *Under Assumptions 1–3, 6B, 7, and 8(a), the value function $V(x, w)$ is identified for all x and w.*

We sketch the proof of Lemma 1 here and leave the details to Appendix. Under Assumption 8(a), we can express the *ex-ante* value function $V(x, w)$ as

$$\begin{aligned} V(x, w) &= -\log p_K(x, w) + v_K(x, w) \\ &= -\log p_K(x, w) + u_K(x, w) + \beta \sum_{x', w'} V(x', w') f(w'|w) s(x'|x, K) \\ &= -\log p_K(x, w) + \beta \sum_{x', w'} V(x', w') f(w'|w) s(x'|x, K). \end{aligned} \tag{20}$$

Intuitively, we could keep iterating this equation to approximate the *ex-ante* value $V(x, w)$ by collecting all of the terms: $-\log p_K(x, w)$, S_K, and $f(w'|w)$, where the first term is observed, and the other two are assumed to be known under Assumptions 6B and 7, respectively. Essentially, the *ex ante* value $V(x, w)$ is a fixed point in the equation above and can be solved directly as a closed-form function of $-\log p_K(x, w)$, S_K, and $f(w'|w)$. Once the *ex-ante* value function is identified, the subjective belief vector $S_i(x)$ is the only unknown in Eq. (19). We show next that $J - 1$ equations in (19) provide sufficient variation for us to identify $S_i(x)$, $i \neq K$.

To better understand the conditions required for identification and to obtain a closed-form expression for subjective beliefs, we stack Eq. (19) for all x and all the $J - 1$ pairs of w into the following matrix equation.

$$\boldsymbol{\Delta\xi}_{i,K}(x) = \beta(S_i(x) - S_K(x))\boldsymbol{V}\Delta\boldsymbol{F}, \tag{21}$$

where $\boldsymbol{\Delta\xi}_{i,K}(x) \equiv [\Delta\xi_{i,K}(x, w^1), \ldots, \Delta\xi_{i,K}(x, w^{J-1})]$ is of dimension $1 \times (J - 1)$, and $\Delta\boldsymbol{F}$, defined analogously to $\Delta\xi_{i,K}$, is of dimension $M \times (J - 1)$.

According to Eq. (21), $S_i(x)$ can be identified if there is sufficient variation in (21), which is guaranteed by the following assumption.

Assumption 5D. The $(J - 1) \times (J - 1)$ matrix $\boldsymbol{V}\Delta\boldsymbol{F}$ has full rank.

This assumption is empirically testable because we can identify the value function from CCPs directly. Assumption 5D is equivalent to assuming that value matrix $\boldsymbol{V}$ and transition matrix $\Delta\boldsymbol{F}$ for the state variable w are full row rank and full column rank, respectively. This requires that both the $J - 1$ rows, $V(x = j, w = k) - V(x = J, w = k)$, for $x = 1, 2, \ldots, J - 1$, $k = 1, 2, \ldots, M$ and the $J - 1$ transition vector, $F(w_1^j) - F(w_2^j)$, for $j = 1, 2, \ldots, J - 1$ are linearly independent. Under this assumption, we can identify the beliefs associated with action i as

$$S_i(x) = S_K(x) + \beta^{-1}\boldsymbol{\Delta\xi}_{i,K}(x)\left[\boldsymbol{V}\Delta\boldsymbol{F}\right]^{-1}. \tag{22}$$

Note that the result above applies to all actions, so we can identify the beliefs associated with other actions analogously. We summarize the results of identification for the infinite horizon case as follows.

Theorem 4. *Suppose that Assumptions 1–3, 5D, 6B, 7–8 are satisfied. Then the subjective beliefs $s(x'|x, a)$ for $x, x' \in \{1, 2, \ldots, J\}$ and $a \in \{1, 2, \ldots, K - 1\}$ are identified as a closed-form function of the CCPs, $p(a|x, w)$, the objective state transition $f(w'|w)$, and the belief of $S_K(x)$.*

Recall that Assumption 7 requires the state transition of w to be independent of agents' action a. This restriction can be relaxed, and Theorem 4 still holds if agent actions do affect the transition of w. The derivation is similar to Theorem 4, so we omit it in the paper.

Remark 2. The identification argument of Theorem 4 can be readily applied to the finite horizon model if $J - 1$ pairs of w exist as in Assumption 8(a), and at least the last two periods of data are observed. Specifically, in finite horizon models, we can follow the identification procedure of Theorem 4 with appropriate adjustments to the assumptions for identifying agent beliefs in two steps. First, we recover *ex-ante* value functions V_t for $t = T - 1$ and T using the recursive relationship under Assumption 8(b) (preference normalization). Second, we use the relationship among CCPs, beliefs, and the identified value function for period $T - 1$, which is similar to Eq. (22), to identify the beliefs. In summary, we only need the last two periods of data to identify agent beliefs, which improves the result in Theorem 3 in terms of data requirements.

Remark 3. In some empirical applications, state variables can be decomposed into states (x) that evolve independently of agents' actions (e.g., GDP), on which agents hold subjective beliefs, and states (w) whose objective transition process depends on agents' actions and is known to agents. One example of the transition process for the latter variable is a deterministic function of past actions and/or states (e.g., experience). In the Appendix we show that if we make these alternative assumptions on state variables instead of Assumption 7, we also can follow the procedure in Theorem 3 to identify the (action-independent) subjective beliefs in finite horizon models. We summarize the result in Theorem A.1 in Appendix. Specifically, we show in Theorem A.1 that: (1) normalization of beliefs such as Assumptions 6A and 6B are not necessary for identification; and (2) the number of periods of data required for identification is $\lceil \frac{J-1}{M-1} \rceil + 2$. These results are improvements over Theorem 3, which is based on Assumption 7. The improvements are due mainly to the exogenous transition of the "macro" state variable. First, when the beliefs evolve endogenously, the useful information for identification is from the relative CCPs to a reference choice, say K. Thus, normalization is necessary. However, in the exogenous case, the beliefs do not depend on any actions and a reference action is no longer necessary. Both Theorems 3 and A.1 exploit the variations of CCPs over time and across the additional state variable w, which greatly reduces data requirements. Nevertheless, exogenous transition of the state variable in Theorem A.1 further reduces the number of periods of data from $\lceil \frac{2J-2}{M} \rceil + 2$ to $\lceil \frac{J-1}{M-1} \rceil + 2$ (note that $\frac{2J-2}{M} \geq \frac{J-1}{M-1}$ for any $M \geq 2$). Specifically, if $M \geq J$, we can identify the beliefs using only three consecutive periods of data.

Remark 4. The result in Remark 3 does not apply to infinite horizon models. In Remark 3, the transition of x is exogenous, so the log ratio of CCPs across actions is affected by the action-dependent transition of w directly and the action-independent beliefs about x indirectly. For infinite horizon models, it is unclear how we can achieve identification under such a setting because the identifying strategies in both Theorems 4 and A.1 fail. First of all, the identification strategy in Theorem 4 relies on the assumption that the subjective beliefs conditional on the reference action K are known. This strategy is infeasible because the subjective beliefs to be identified $s(x'|x)$ are now action-independent. Second, the approach in Theorem A.1 is not applicable because there does not exist a recursive relationship for the value function $\boldsymbol{V}$ due to its stationarity in infinite horizon models.

5. An Extension: Heterogeneous beliefs

Agents may have heterogeneous preferences and/or beliefs about the transition of the same state variable. We prove in this section that our identification results are applicable to a DDC model with such heterogeneity.

Suppose agents can be classified into $H \geq 2$ types, and H is known to the econometrician.[7] Let $\tau \in \{1, 2, \ldots, H\}$ denote the unobserved type (heterogeneity). All agents of the same type have the same subjective beliefs and preferences, denoted as $s(x'|x, a, \tau)$ and $u_a(x, \tau)$, respectively. Similarly, the CCP for agents of type τ in period t is $p_t(a|x, \tau)$. An agent's type is assumed to be time invariant. We utilize an identification methodology developed in the measurement error literature i.e., Hu (2008), to prove that the observed joint distribution of state variables and agent actions uniquely determine the type-specific CCP $p_t(a|x, \tau)$ for all t and $\tau \in \{1, 2, \ldots, H\}$. We then apply the results in Sections 3–4 to identify the heterogeneous beliefs $s(x'|x, a, \tau)$ and utility functions $u_a(x, \tau)$ associated with type τ. We present our analysis for the finite horizon models, while the result is readily applicable to the infinite horizon case.

We start from the observed CCP $p_t(a|x)$, which can be expressed as a weighted average of the H components $p_t(a|x, \tau)$.

$$p_t(a|x) = \sum_{\tau=1}^{H} p_t(a|x, \tau)q(\tau), \tag{23}$$

where we assume $q(\tau|x) = q(x)$, i.e., the distribution of type τ is predetermined and does not depend on x. Hu (2008) shows that the unknowns on the right-hand-side of the equation can be nonparametrically identified if there exist two measurements of the latent type variable τ and a (binary) variable correlated with τ. In our DDC model, we use actions as the measurements of τ and the state variable as the one correlated with τ. In the rest of this section, we present all the assumptions under which the CCPs $p_t(a|x, \tau)$ are identified and leave the proof in Appendix.[8]

Assumption 9 (*Markov Property*)**.** For any given type τ, $\{a_t, x_t\}$ follows a first-order Markov process.

The first-order Markov property of actions and state variables is widely assumed in the literature of DDC models. A sufficient condition of this assumption is that the state variable evolves through a first-order Markov process, and an agent's decision depends only on the current state. Under Assumption 9, the observed joint distribution of the state

[7] The number of types H may be inferred from the data, see e.g., Kasahara and Shimotsu (2009). For ease of exposition, we assume H to be known.

[8] The finite mixture model in Eq. (23) can also be identified using the methodology proposed in Kasahara and Shimotsu (2009). Our method of identification relies on variation of actions while Kasahara and Shimotsu (2009) utilize variation of the state variable x.

variables and actions is associated with the unobserved types as follows.

$$\begin{aligned}
&\Pr(a_{t+l},\ldots,a_{t+1},x_{t+1},a_t,x_t,a_{t-1},\ldots,a_{t-l})\\
&=\sum_{\tau=1}^{H}\Pr(a_{t+l},\ldots,a_{t+1}|x_{t+1},\tau)\Pr(x_{t+1},a_t|x_t,\tau)\Pr(\tau,x_t,a_{t-1},\ldots,a_{t-l})\\
&=\sum_{\tau=1}^{H}\Pr(a_{t+l},\ldots,a_{t+1}|x_{t+1},\tau)\Pr(a_t|x_{t+1},x_t,\tau)\Pr(x_{t+1}|x_t,\tau)\Pr(\tau,x_t,a_{t-1},\ldots,a_{t-l}).
\end{aligned}$$

In the equation above, the joint distribution of states and actions is expressed as a misclassification model (Hu, 2008), with the vectors of actions $(a_{t+l},\ldots,a_{t+1})$ and $(a_{t-1},\ldots,a_{t-l})$ being two measurements for the unobserved types τ. We require the integer l to satisfy the inequality $H \le K^l$, i.e., the number of possible values the two vectors $(a_{t+l},\ldots,a_{t+1})$ and $(a_{t-1},\ldots,a_{t-l})$ take is no smaller than the number of types, H. If the latent variable takes more values than its measurements do, then the model is generally not identifiable.

To apply the identification strategy of eigenvalue–eigenvector decomposition from the measurement error literature, e.g., Hu (2008), we reduce the number of values $(a_{t+l},\ldots,a_{t+1})$ and $(a_{t-1},\ldots,a_{t-l})$ take to be the same as the number of latent types through a known function $h(\cdot)$. Specifically, function $h(\cdot)$ maps the support of $(a_{t+l},\ldots,a_{t+1})$ and $(a_{t-1},\ldots,a_{t-l})$, to that of τ. We define the two measurements for the latent type as,

$$\begin{aligned}
a_{t+} &\equiv h(a_{t+l},\ldots,a_{t+1}),\\
a_{t-} &\equiv h(a_{t-1},\ldots,a_{t-l}).
\end{aligned} \tag{24}$$

We exemplify function $h(\cdot)$ as follows. Suppose that we consider a dynamic investment problem where agents choose whether (or not) to invest in a stock ($a_t \in \{1, 0\}$, $a_t = 1$ and $a = 0$ stand for "invest" and "do not invest", respectively). An agent's decision depends on both a discrete state variable x_t that describes her wealth and her subjective beliefs about wealth transition based on the return of stock. For illustrative purposes, we assume that all the agents in our analysis have homogeneous preference but heterogeneous beliefs. The unobserved type captures the accuracy of their subjective beliefs, i.e., how close they are from the *ex-post* distribution of returns. The type takes three values: "more accurate", "accurate", and "less accurate", denoted as $\tau = 1, 2,$ and 3, respectively. In this setting, $H = 3$, and it is sufficient to choose $l = 2$ such that $H = 3 < K^l = 4$. The function $h(\cdot)$ maps the support of actions in two consecutive periods, i.e., $\{(0,0),(0,1),(1,0),(1,1)\}$, to the space of the unobserved types, i.e., $\{1, 2, 3\}$. One possible choice of $h(\cdot)$ is

$$a_{t+} = \begin{cases} 1, & \text{if } (a_{t+1},a_{t+2}) = (0,0), \text{ invest in neither periods}\\ 2, & \text{if } (a_{t+1},a_{t+2}) = (0,1) \text{ or } (a_{t+1},a_{t+2}) = (1,0), \text{ invest in one of the two periods}\\ 3, & \text{if } (a_{t+1},a_{t+2}) = (1,1), \text{ invest in both periods.}\end{cases}$$

The other measurement a_{t-} can be defined analogously.[9]

For a given pair $(x_t, x_{t+1}) \in \mathcal{X}\times\mathcal{X}$, we define a matrix to summarize the joint distribution of a_{t+} and a_{t-}.

$$M_{a_{t+},x_{t+1},x_t,a_{t-}} = \left[\Pr(a_{t+}=i, x_{t+1}, x_t, a_{t-}=j)\right]_{i,j}.$$

Our identification requires that the H by H matrix $M_{a_{t+},x_{t+1},x_t,a_{t-}}$ has full rank.

Assumption 10. For all $(x_{t+1}, x_t) \in \mathcal{X}\times\mathcal{X}$, the rank of the matrix $M_{a_{t+},x_{t+1},x_t,a_{t-}}$ is H.

In the investment example above, this assumption requires that the three columns of the 3 by 3 matrix $M_{a_{t+},x_{t+1},x_t,a_{t-}}$ are linearly independent for any given x_t and x_{t+1}. Assumption 10 is empirically testable because it is imposed on a matrix that is directly estimable from the data. Moreover, we have the flexibility to choose $h(\cdot)$ such that the assumption holds. This is because the mapping is not unique and the rank of the matrix depends on the construction of a_{t+} and a_{t-} through the mapping.

Following Hu (2008), our identification strategy involves an eigenvalue–eigenvector decomposition of an estimable matrix, and such a decomposition must be unique for the purpose of identification. The uniqueness requires the eigenvalues of the decomposition to be distinct. That is guaranteed by the following assumption.

Assumption 11 (*Distinct Eigenvalues*)**.** For any given $(x_{t+1}, x_t) \in \mathcal{X}\times\mathcal{X}$, there exists a choice $k \in \mathcal{A}$ such that $\Pr(a_t = k|x_{t+1}, x_t, \tau)$ differs for any two different types.

To investigate the restrictions Assumption 11 imposes to our model, we write the probability $\Pr(a_t = k|x_{t+1}, x_t, \tau)$ as:

$$\Pr(a_t = k|x_{t+1},x_t,\tau) = \frac{\Pr(x_{t+1}|x_t,a_t=k)\Pr(a_t=k|x_t,\tau)}{\sum_{a_t\in\mathcal{A}}\Pr(x_{t+1}|x_t,a_t)\Pr(a_t|x_t,\tau)}.$$

[9] It is clear from the example that the choice of $h(\cdot)$ is not unique. As we show in the Appendix, given the assumptions of identification hold, the choice of $h(\cdot)$ does not affect our estimation results.

The equality holds because the transition of x given action a_t, $\Pr(x_{t+1}|x_t, a_t)$, does not depend on agent type τ. An important implication of the equation above is that if agent actions are binary, then a sufficient condition of Assumption 11 is that the CCP $\Pr(a_t = k|x_t, \tau)$ differs for any two different types. In the investment example discussed above, this sufficient condition requires that, given financial status in period t, agents with different types would choose to invest (or not) with different probabilities. This requirement indicates that beliefs of different types need to differ sufficiently such that given the state variable their CCPs are distinct. It is worth noting that the assumption is required to hold only for an action k rather than all actions.

The uniqueness of the eigenvalue–eigenvector decomposition also requires that we correctly order all the eigenvalues. This can be achieved under the assumption below.

Assumption 12 (*Monotonicity*). For any given $x_{t+1} \in \mathcal{X}$, there exists a known $m \in \{1, 2, \ldots, K^l\}$ such that $\Pr(a_{t+} = m|x_{t+1}, \tau)$ is strictly monotonic in τ.

Let us interpret the restriction this assumption imposes to the model by using the investment example again. Suppose that $a_{t+} = (1, 1)$ satisfies the assumption: $\Pr(a_{t+} = (1, 1)|x_{t+1}, \tau)$ is strictly decreasing in τ, then it implies that the more accurate the subjective beliefs, the higher the probability with which agents choose to invest in both periods of $t + 1$ and $t + 2$. Recall that a_{t+} is defined as $h(a_{t+l}, \ldots, a_{t+1})$ and the choice of $h(\cdot)$ is not unique. This gives us some flexibility to choose $h(\cdot)$ such that Assumption 12 holds. The probability $\Pr(a_{t+} = m|x_{t+1}, \tau)$ is different from a CCP, thus Assumption 11 is neither sufficient nor necessary for Assumption 12. In empirical applications, the sufficient conditions of these two assumptions are often model specific, e.g., see similar assumptions in An et al. (2010) and An (2017).

We summarize the result of identification in the following theorem.

Theorem 5. *If Assumptions 9–12 hold, then the type-specific CCPs, $p_t(a_t|x_t, \tau)$, are uniquely determined by the joint distribution $\Pr(a_{t+l}, \ldots, a_t, x_{t+1}, x_t, a_{t-1}, \ldots, a_{t-l})$, where $H \le K^l$.*

Once type-specific CCPs $p_t(a_t|x_t, \tau)$ are identified, we can proceed to identify both utility and subjective beliefs for each type of agents using the results in Sections 3 and 4. The heterogeneity of agents can be in one of the three scenarios: they have different subjective beliefs, or preferences, or both. Our identification procedure allows us to recover agent utility functions and subjective beliefs for each type, thus we are able to distinguish the three scenarios.

6. Estimation and Monte Carlo evidence

In this section, we first discuss the estimation of DDC models with subjective beliefs. Then we present some Monte Carlo evidence for our proposed estimators.

6.1. Estimation

Our identification result provides a closed-form solution to the agent subjective beliefs for both finite and infinite horizon models. One may follow the identification procedure to estimate the subjective beliefs by using a closed-form estimator. Agent preferences then can be estimated in a second step using the CCP approach based on Hotz and Miller (1993). However, such a closed-form estimator involves inversion of matrices, thus its performance would be unstable if the matrices are near singular. Alternatively, we propose a maximum likelihood estimator to estimate subjective beliefs and agent preferences in one step.

Suppose that we observe in the data n agent actions for T periods, together with the states, denoted as $\{a_{it}, x_{it}\}$, $i = 1, \ldots, n$, $t = 1, \ldots, T$. We denote the parameters in utility functions, objective transitions, and subjective beliefs as θ_u, θ_o, and θ_s, respectively. We first present the likelihood function of the data $\{a_{it}, x_{it}\}$.

$$\begin{aligned}&\mathcal{L}(x_2, \ldots, x_T, a_1, \ldots, a_T|x_1; \theta_u, \theta_s, \theta_o)\\ =&\prod_{i=1}^{n}\prod_{t=2}^{T} p_t(a_{it}|x_{it}; \theta_u, \theta_s) f(x_{it}|x_{i,t-1}, a_{it-1}; \theta_o) p_1(a_{i1}|x_{i1}; \theta_u, \theta_s),\end{aligned}$$

where $p_t(a_{it}|x_{it}; \theta_u, \theta_s)$ is agent i's CCPs in period t. The log-likelihood function is additively separable:

$$L \equiv \log \mathcal{L} = \sum_{i=1}^{n}\sum_{t=1}^{T} \log p_t(a_{it}|x_{it}; \theta_u, \theta_s) + \sum_{i=1}^{n}\sum_{t=2}^{T} \log f(x_{it}|x_{it-1}, a_{it-1}; \theta_o). \tag{25}$$

Thus, we can estimate preferences θ_u and subjective beliefs θ_s separately from objective transition θ_o. That is, the parameters (θ_u, θ_s) and θ_o can be estimated by maximizing the first and second parts of the log-likelihood function above, respectively. We use $\hat{\theta}$ to denote the estimator of the corresponding parameters θ.

Recall that some elements of subjective beliefs have to be normalized for identification. Without loss of generality, we assume that agents have rational expectations about some state transition so that we can use the objective transition of these states in estimation. Under such normalization, we divide the parameters associated with subjective beliefs into two parts: $\theta_s \equiv (\theta_s^n, \theta_s^e)$, where θ_s^n and θ_s^e are the parameters to be normalized and estimated, respectively. θ_s^n can be obtained from the estimated objective state transition $\hat{\theta}_o$, denoted as $\hat{\theta}_s^n$. The parameters of preferences and subjective beliefs, $\theta \equiv (\theta_u, \theta_s^e)$, can be estimated through the following maximization,

$$\max_{\theta} \sum_{i=1}^{n} \sum_{t=1}^{T} \log p_t(a_{it}|x_{it}; \theta, \hat{\theta}_s^n). \tag{26}$$

Because the CCP $p_t(a_{it}|x_{it}; \theta, \theta_s^n)$ is solved differently for finite horizon models from infinite horizon models, we present the estimators for the two scenarios separately.

Finite horizon. For a finite horizon model, we use backward induction to solve for the CCPs for each period t. To do so, we start from the terminal period T, in which the optimal behaviors depend on the continuation value. If we assume that the continuation value is zero in the terminal period, then the choice-specific value function is the same as the per-period flow utility. Next, we proceed to period $T-1$ and continue the procedure till we reach the first period.

Infinite horizon. In the case of infinite horizon, the dynamic framework is stationary, i.e., the value function and CCPs do not change over time. For a given set of CCPs, the *ex-ante* value function can be solved as a fixed-point of a system of equations. Plugging the *ex-ante* value function into Eq. (2) for all actions and states, we can further represent the CCPs as a fixed point in the following mapping,

$$p = \Psi(p; \theta_u, \theta_s), \tag{27}$$

where p, a vector of $J(K+1) \times 1$, collects CCPs for all actions and states, and $\Psi(\cdot)$ represents the fixed point mapping of this vector.

To estimate θ, we adopt a Nested Pseudo Likelihood Algorithm (NPL) proposed in Aguirregabiria and Mira (2002). To implement the algorithm, we start with an initial guess $p^{(0)}$ for the CCPs. In the mth ($m \geq 1$) step of iteration, our estimation takes the following two steps.

- Step 1: Given $p^{(m-1)}$, we obtain a pseudo-likelihood estimate of θ, $\hat{\theta}^{(m)}$, which satisfies
$$\hat{\theta}^{(m)} = \arg\max_{\theta} \sum_{i=1}^{n} \sum_{t=1}^{T} \log p^{(m)}(a_{it}|x_{it}; \theta, \hat{\theta}_s^n),$$
where $p^{(m)}(a_{it}|x_{it}; \theta)$ is an element of $p^{(m)}$ satisfying the mapping $p^{(m)} = \Psi(p^{(m-1)}; \theta, \hat{\theta}_s^n)$.
- Step 2: We update the CCPs by plugging the estimated parameters $\hat{\theta}^{(m)}$ into the mapping
$$p^{(m)} = \Psi(p^{(m-1)}; \theta^{(m)}, \hat{\theta}_s^n).$$

We iterate these two steps till both p and θ converge. We refer to Kasahara and Shimotsu (2008) for convergence of the estimator generated from the NPL algorithm to the Maximum likelihood Estimation (MLE). In particular, the NPL estimator converges to the MLE estimator at a super-linear, but less-than-quadratic, rate.

The difference between our estimator and those in the existing literature lies in the role of subjective beliefs in the estimation. Specifically, we estimate part of the subjective beliefs θ_s^e together with the payoff primitives θ_u. By contrast, the existing literature assumes that the subjective beliefs are the same as the objective state transitions, i.e., $\theta_s = \theta_o$, and then estimates θ_s directly from data in the first step and the payoff primitives θ_u in the second step.

Heterogeneous subjective beliefs. In the case of heterogeneous subjective beliefs (Section 5), we can apply the EM algorithm proposed in Arcidiacono and Miller (2011) to estimate type-specific preferences and subjective beliefs, as well as the type probabilities.

Suppose that the number of types H is known to the econometrician. Let $\theta_u \equiv (\theta_u^1, \ldots, \theta_u^H)$ and $\theta_s \equiv (\theta_s^1, \ldots, \theta_s^H)$, where θ_u^τ and θ_s^τ denote the parameters of preferences and beliefs for type τ, respectively, and θ_o is defined the same as before. Let $q(\tau)$ be the population probability of being type τ such that $\sum_{\tau=1}^{H} q(\tau) = 1$, and $q \equiv (q(1), q(2), \ldots, q(H))$. The likelihood function of the data can be represented as

$$\begin{aligned}
&\mathcal{L}(x_2, \ldots, x_T, a_1, \ldots, a_T | x_1; \theta_u, \theta_s, \theta_o) \\
&= \prod_{i=1}^{n} \left(\sum_{\tau=1}^{H} q(\tau) \prod_{t=2}^{T} p_t(a_{it}|x_{it}; \theta_u^\tau, \theta_s^\tau) f(x_{it}|x_{i,t-1}, a_{it-1}; \theta_o) p_1(a_{i1}|x_{i1}; \theta_u^\tau, \theta_s^\tau) \right) \\
&= \prod_{i=1}^{n} \left(\prod_{t=2}^{T} f(x_{it}|x_{i,t-1}, a_{it-1}; \theta_o) \sum_{\tau=1}^{H} q(\tau) \prod_{t=1}^{T} p_t(a_{it}|x_{it}; \theta_u^\tau, \theta_s^\tau) \right).
\end{aligned}$$

Let θ_s^n and θ_s^e be parameters of beliefs to be normalized and estimated, respectively. Let $\theta \equiv \{\theta_u, \theta_s^e\}$. Similar to the case of homogeneous subjective beliefs, θ_s^n can be obtained from the estimated objective state transition $\hat{\theta}_o$, which is the same across all types.

Given the estimate $\hat{\theta}_s^n$, we can estimate the parameters θ and q using the EM algorithm as in Arcidiacono and Miller (2011). Let $\pi_{i\tau}$ denote the probability agent i is of type τ. In the mth iteration, given estimates $\theta^{(m-1)}$ and $q^{(m-1)}$, we first update $\pi_{i\tau}^{(m)}$,

$$\pi_{i\tau}^{(m)} = \frac{\hat{q}^{(m-1)}(\tau)\prod_{t=1}^{T} p_t(a_{it}|x_{it};\theta^{(m-1)},\hat{\theta}_s^n)}{\sum_{\tau'=1}^{H}\hat{q}^{(m-1)}(\tau')\prod_{t=1}^{T} p_t(a_{it}|x_{it};\theta^{(m-1)},\hat{\theta}_s^n)}. \tag{28}$$

The type probabilities are updated as:

$$q^{(m)}(\tau) = \frac{1}{n}\sum_{i=1}^{n}\pi_{i\tau}^{(m)}. \tag{29}$$

Finally, we obtain $\theta^{(m)}$ by maximizing the following log likelihood function,

$$\theta^{(m)} = \max_{\theta}\sum_{i=1}^{n}\sum_{\tau=1}^{H}\sum_{t=1}^{T}\pi_{i\tau}^{(m)}\log\left(p_t(a_{it}|x_{it};\theta_\tau,\hat{\theta}_s^n)\right). \tag{30}$$

We iterate the steps in Eqs. (28)–(30) till both θ and q converge.

6.2. Monte Carlo experiments

In this section, we present some Monte Carlo results to illustrate the finite sample performance of the proposed estimators. The Monte Carlo experiments are conducted for both finite and infinite horizon models with homogeneous beliefs.

We consider a binary choice DDC model in both finite and infinite horizon scenarios. First, we set up the payoff primitives, the objective law of motion, and agent beliefs about the transition of the state variable. Given these primitives, we solve for agent CCPs by backward induction and contraction mapping in the finite and infinite models, respectively. We then use the CCPs and objective transition matrices to simulate agent actions and states. Next, we estimate the parameters of interest, following the proposed procedure of estimation. The objective transition matrices are estimated using MLE, and the payoff primitives and subjective beliefs are estimated together using MLE and NPL estimators in finite and infinite horizon cases, respectively.

In the finite horizon case, the per-period utility function is specified as follows.

$$u(a,x) = \begin{cases}\epsilon_0, & \text{if } a=0,\\ u(x)+\epsilon_1, & \text{if } a=1,\end{cases}$$

where ϵ_0 and ϵ_1 are drawn from a mean-zero type-I extreme value distribution. Note that the continuation value at the terminal period is assumed to be zero. We set $J=3$, i.e., the state variable x takes three values, $x\in\{x_1,x_2,x_3\}$, so we have three utility parameters: $u(x_1)=-2$, $u(x_2)=0.4$, and $u(x_3)=2.1$. The objective transition for the state variable x conditional on choice $a=0$ and $a=1$, respectively, is represented by the following 3×3 matrices

$$\boldsymbol{TR}_0 = \begin{bmatrix}0.8 & 0.1 & 0.1\\ 0.2 & 0.6 & 0.2\\ 0.1 & 0.19 & 0.71\end{bmatrix}; \qquad \boldsymbol{TR}_1 = \begin{bmatrix}0.2 & 0.6 & 0.2\\ 0.5 & 0.2 & 0.3\\ 0.2 & 0.3 & 0.5\end{bmatrix}.$$

Let $\boldsymbol{S}_0$ and $\boldsymbol{S}_1$ denote subjective beliefs on the transition conditional on the action 0 and 1, respectively. We consider three different scenarios for the beliefs: (1) Agents have rational expectations, i.e., $\boldsymbol{S}_1=\boldsymbol{TR}_1$ and $\boldsymbol{S}_0=\boldsymbol{TR}_0$. (2) Agents' subjective beliefs about the state transition conditional on action $a=1$ are the same as their objective counterparts, i.e., $\boldsymbol{S}_1=\boldsymbol{TR}_1$; agents' subjective beliefs about the state transition conditional on action $a=0$ deviates from their objective counterparts:

$$\boldsymbol{S}_0 = \begin{bmatrix}0.9 & 0.05 & 0.05\\ 0.1 & 0.8 & 0.1\\ 0.05 & 0.095 & 0.855\end{bmatrix}.$$

(3) Agents' subjective beliefs on the transition of one state $x=J$ conditional on action $a=1$ are the same as their objective counterparts, i.e., $\boldsymbol{S}_1(3)=\boldsymbol{TR}_1(3)$; the beliefs on the rest of the transition deviate from their counterparts and are expressed as follows:

$$\boldsymbol{S}_0 = \begin{bmatrix}0.9 & 0.05 & 0.05\\ 0.1 & 0.8 & 0.1\\ 0.05 & 0.095 & 0.855\end{bmatrix}, \qquad \boldsymbol{S}_1 = \begin{bmatrix}0.6 & 0.3 & 0.1\\ 0.25 & 0.6 & 0.15\\ 0.2 & 0.3 & 0.5\end{bmatrix}.$$

Settings (2) and (3) satisfy Assumptions 6A and 6B, respectively. In these two settings, identification requires $J+1=4$ and $2*J=6$ periods of data, respectively. For comparability of estimator performance across the two settings, we simulate the data for 6 periods, regardless of the setting.

For the infinite horizon setting, the agent choice is binary, and there are two state variables, x and w. Based on the identification results, the payoff function is assumed as follows

$$u(a, x, w) = \begin{cases} \epsilon_0, & \text{if } a = 0; \\ u^1(x) + u^2(x)w + \epsilon_1, & \text{if } a = 1, \end{cases} \tag{31}$$

where ϵ_0 and ϵ_1 are drawn from a mean-zero type-I extreme value distribution. Both state variables are assumed to be discrete: $x \in \{x_1, x_2\}$ and $w \in \{w_1, w_2, w_3, w_4\}$. The utility parameters are $u^1(x_1) = 0.1$, $u^1(x_2) = 0.2$, $u^2(x_1) = 0.2$, and $u^2(x_2) = -0.2$. The objective state transition processes for x, $\boldsymbol{TR}_0^x$, and $\boldsymbol{TR}_1^x$, and for w, $\boldsymbol{TR}_0^w$, and $\boldsymbol{TR}_1^w$, are setup as follows.

$$\boldsymbol{TR}_0^w = \begin{bmatrix} 0.6 & 0.2 & 0.2 & 0 \\ 0.1 & 0.75 & 0.15 & 0 \\ 0.04 & 0.1 & 0.8 & 0.06 \\ 0.01 & 0.08 & 0.1 & 0.81 \end{bmatrix}; \quad \boldsymbol{TR}_1^w = \begin{bmatrix} 0.7 & 0.1 & 0.15 & 0.05 \\ 0.2 & 0.65 & 0.05 & 0.1 \\ 0.04 & 0.01 & 0.9 & 0.05 \\ 0.02 & 0.18 & 0.1 & 0.7 \end{bmatrix};$$

$$\boldsymbol{TR}_0^x = \begin{bmatrix} 0.6 & 0.4 \\ 0.45 & 0.55 \end{bmatrix}; \quad \boldsymbol{TR}_1^x = \begin{bmatrix} 0.1 & 0.9 \\ 0.5 & 0.5 \end{bmatrix},$$

where the subscripts 0 and 1, respectively, represent $a = 0$ and $a = 1$.

Agents have rational expectations about the transition of w but may have subjective beliefs about the transition of x. We consider two settings for Monte Carlo experiments: (1) Agents have rational expectations; that is, agent beliefs about the state evolution are the same as their objective counterparts, i.e., $\boldsymbol{S}_a^w = \boldsymbol{TR}_a^w$, $\boldsymbol{S}_a^x = \boldsymbol{TR}_a^x$, $a \in \{0, 1\}$. (2) Agents' subjective beliefs satisfy Assumption 6B, and $\boldsymbol{S}_a^w = \boldsymbol{TR}_a^w$, $a \in \{0, 1\}$ and $\boldsymbol{S}_1^x = \boldsymbol{TR}_1^x$, while $\boldsymbol{S}_0^x \neq \boldsymbol{TR}_0^x$, where

$$\boldsymbol{S}_0^x = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}.$$

In each of the scenarios, we use sample $n = 300, 600, 1000$, and 2500, and standard errors are computed from 1000 replications. Before estimation, we check the full rank Assumptions 4 and 5A for our simulated samples and find that the assumptions hold.

The results of the Monte Carlo experiments for the finite horizon case are presented in Tables 1–4, and for the infinite horizon case in Tables 5–6. We draw two main findings. First, the proposed estimator performs well across different settings for moderate sample sizes. More importantly, as shown in Tables 1, 3, and 5, our estimates track the true parameters closely, even when the data are generated from a model with rational expectations. Not surprisingly, in the cases where agents have rational expectations, the standard errors of our estimates are generally larger than those from imposing the restriction of rational expectations. Second, failing to account for subjective beliefs may lead to significant estimation bias. When data are generated from subjective beliefs, the parameters of the utility function estimated by imposing rational expectations are off the true values. This can be seen in Tables 2 and 4, where estimates of $u(x_1)$, $u(x_2)$, and $u(x_3)$ are dramatically different from the true parameters. The differences persist as sample sizes increase from 300 to 2500.

7. Empirical illustration: Women's labor force participation

Now we apply the proposed method to the Panel Study of Income Dynamics (PSID) data and focus on women's labor force participation. Female labor supply has been studied extensively in the literature; e.g., see Eckstein and Wolpin (1989) and Blundell et al. (2016), among others. Instead of providing a thorough analysis of women's labor force participation, simply this section is an illustrative application of our identification and estimation results.

We assume that a woman and her husband jointly make a decision on her labor force participation. Their beliefs about the evolution of their household income affect the wife's labor participation decisions. Our main objectives here are: (1) to investigate whether their beliefs deviate from rational expectations; and if yes, (2) to conduct a counterfactual analysis, analyzing how women's decisions would change if they were instead to have rational expectations.

We follow Eckstein and Wolpin (1989) in making some key working assumptions in this analysis. First, we simplify the choice of hours of work to a binary working/non-working decision. Second, we ignore the husband's labor force participation decisions. Third, we only consider women older than 38 in order to avoid modeling fertility decisions. Finally, we take marriage as exogenously given.

7.1. Data

The PSID is a longitudinal survey consisting of a nationally representative sample of over 18,000 individuals living in 5000 families in the United States. The original sample was re-interviewed annually from 1968 to 1997 and biennially thereafter. The PSID collects data on annual income and female labor force participation for the preceding calendar year. We only use data collected up to 1997, because our identification strategy relies on variations in CCPs over consecutive years.

We construct an annual employment profile for each woman between the age of 39 and 60, where 60 is assumed to be the terminal period of a woman's labor participation decision: women are typically out of the labor force by age 60,

Table 1
Simulation results for a data generating process (DGP) of rational expectations (RE): finite horizon.

		Estimates with SB				Estimates with rational expectations			
	True	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$
u_1	−2	−2.01	−2.01	−2.00	−2.00	−2.00	−2.00	−2.00	−2.00
		(0.25)	(0.19)	(0.14)	(0.09)	(0.14)	(0.10)	(0.08)	(0.05)
u_2	0.4	0.39	0.40	0.39	0.40	0.40	0.40	0.40	0.40
		(0.23)	(0.16)	(0.13)	(0.08)	(0.15)	(0.10)	(0.08)	(0.05)
u_3	2.1	2.15	2.14	2.12	2.11	2.10	2.10	2.10	2.10
		(0.37)	(0.26)	(0.19)	(0.12)	(0.20)	(0.14)	(0.11)	(0.07)
$\boldsymbol{S}_0(1\|1)$	0.8	0.74	0.74	0.76	0.78				
		(0.26)	(0.24)	(0.20)	(0.15)				
$\boldsymbol{S}_0(2\|1)$	0.1	0.16	0.17	0.17	0.14				
		(0.21)	(0.20)	(0.18)	(0.13)				
$\boldsymbol{S}_0(3\|1)$	0.1	0.10	0.09	0.08	0.08				
		(0.14)	(0.12)	(0.09)	(0.07)				
$\boldsymbol{S}_0(1\|2)$	0.2	0.22	0.21	0.21	0.21				
		(0.31)	(0.29)	(0.27)	(0.23)				
$\boldsymbol{S}_0(2\|2)$	0.6	0.43	0.45	0.46	0.50				
		(0.35)	(0.35)	(0.33)	(0.32)				
$\boldsymbol{S}_0(3\|2)$	0.2	0.35	0.35	0.33	0.29				
		(0.32)	(0.31)	(0.28)	(0.23)				
$\boldsymbol{S}_0(1\|3)$	0.1	0.10	0.09	0.09	0.09				
		(0.12)	(0.10)	(0.10)	(0.08)				
$\boldsymbol{S}_0(2\|3)$	0.19	0.21	0.20	0.20	0.20				
		(0.28)	(0.23)	(0.21)	(0.15)				
$\boldsymbol{S}_0(3\|3)$	0.71	0.69	0.71	0.71	0.71				
		(0.24)	(0.19)	(0.16)	(0.10)				
$\boldsymbol{S}_1(1\|1)$	0.2	0.32	0.31	0.29	0.30				
		(0.31)	(0.29)	(0.27)	(0.24)				
$\boldsymbol{S}_1(2\|1)$	0.6	0.39	0.41	0.46	0.47				
		(0.38)	(0.37)	(0.36)	(0.34)				
$\boldsymbol{S}_1(3\|1)$	0.2	0.30	0.28	0.25	0.24				
		(0.29)	(0.24)	(0.20)	(0.16)				
$\boldsymbol{S}_1(1\|2)$	0.5	0.39	0.39	0.40	0.41				
		(0.36)	(0.32)	(0.30)	(0.24)				
$\boldsymbol{S}_1(2\|2)$	0.2	0.24	0.24	0.25	0.27				
		(0.32)	(0.30)	(0.30)	(0.27)				
$\boldsymbol{S}_1(3\|2)$	0.3	0.37	0.37	0.35	0.33				
		(0.34)	(0.31)	(0.28)	(0.21)				

Note: GDP is under rational expectations: $\boldsymbol{S}_0 = \boldsymbol{T}_0$ and $\boldsymbol{S}_1 = \boldsymbol{T}_1$.
Estimation with subjective beliefs is under the normalization $\boldsymbol{S}_1(x_3) = \boldsymbol{T}_1(x_3)$.

e.g., see Eckstein and Wolpin (1989). The number of years observed in the data varies across women. Table 7 summarizes the frequency of observations by years. As shown there, the 1673 women in our sample are not evenly distributed across years. About 34% of them appear in 17 years of data; more than half of them appear in over 13 years of data. Table 8 presents the summary statistics of our sample. We aggregated the information for those women who are observed at least six years (for the purpose of identification), and income is expressed in 1999 dollars. The average household income was $57,700, with relatively large variation across households and over years. The majority of the women in our sample were between 42 and 55 years of age, and their average educational attainment was high school. On average (at the individual-by-year level), 58% of these women were employed at the time of the survey.

7.2. Model specification

Each household in the sample is assumed to maximize its present value of utility over a known finite horizon by choosing whether the wife works in each discrete time period. The discount factor β is assumed to be 0.95. This framework fits into a finite horizon model. A household's flow utility function is assumed to be stationary and is specified as (we suppress the index t whenever there is no ambiguity)

$$u(a, x, \epsilon) = \begin{cases} \epsilon_0, & \text{if } \ a = 0, \\ u(x) + \epsilon_1, & \text{if } \ a = 1, \end{cases}$$

where ϵ_0 and ϵ_1 are drawn from a mean-zero type-I extreme value distribution and are assumed to be independent over time; a is a binary variable that equals 1 if the wife works, and 0 otherwise; x is the household income.

Analysis of finite horizon DDCs requires assumptions of continuation values in the terminal period. In the existing literature, a simple approach is to assume the continuation value in the terminal period to be zero, then the choice-specific value function in the terminal period is the same as the stationary flow utility. This assumption simplifies estimation:

Table 2
Simulation results for a DGP of subjective beliefs (SB): finite horizon.

	True	Estimates with subjective beliefs				Estimates with rational expectations			
		$N = 300$	$N = 600$	$N = 1000$	$N = 2500$	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$
u_1	−2	−2.03	−2.02	−2.00	−2.00	−2.29	−2.28	−2.28	−2.28
		(0.24)	(0.17)	(0.13)	(0.09)	(0.14)	(0.10)	(0.08)	(0.05)
u_2	0.4	0.40	0.40	0.39	0.40	0.47	0.47	0.47	0.47
		(0.24)	(0.17)	(0.13)	(0.08)	(0.14)	(0.10)	(0.08)	(0.05)
u_3	2.1	2.13	2.13	2.12	2.11	1.66	1.66	1.66	1.66
		(0.34)	(0.24)	(0.19)	(0.12)	(0.16)	(0.12)	(0.09)	(0.06)
$\boldsymbol{S}_0(1\|1)$	0.9	0.86	0.86	0.88	0.90				
		(0.20)	(0.19)	(0.16)	(0.12)				
$\boldsymbol{S}_0(2\|1)$	0.05	0.08	0.09	0.07	0.05				
		(0.16)	(0.15)	(0.13)	(0.08)				
$\boldsymbol{S}_0(3\|1)$	0.05	0.06	0.05	0.05	0.05				
		(0.10)	(0.09)	(0.07)	(0.05)				
$\boldsymbol{S}_0(1\|2)$	0.1	0.16	0.14	0.15	0.13				
		(0.25)	(0.23)	(0.23)	(0.19)				
$\boldsymbol{S}_0(2\|2)$	0.8	0.57	0.60	0.62	0.65				
		(0.36)	(0.34)	(0.33)	(0.31)				
$\boldsymbol{S}_0(3\|2)$	0.1	0.27	0.27	0.24	0.22				
		(0.29)	(0.27)	(0.23)	(0.22)				
$\boldsymbol{S}_0(1\|3)$	0.05	0.04	0.04	0.04	0.04				
		(0.06)	(0.05)	(0.04)	(0.03)				
$\boldsymbol{S}_0(2\|3)$	0.095	0.12	0.10	0.10	0.10				
		(0.19)	(0.14)	(0.11)	(0.09)				
$\boldsymbol{S}_0(3\|3)$	0.855	0.84	0.86	0.86	0.86				
		(0.19)	(0.13)	(0.10)	(0.08)				
$\boldsymbol{S}_1(1\|1)$	0.6	0.55	0.56	0.57	0.62				
		(0.32)	(0.31)	(0.28)	(0.22)				
$\boldsymbol{S}_1(2\|1)$	0.3	0.32	0.34	0.34	0.28				
		(0.33)	(0.32)	(0.31)	(0.25)				
$\boldsymbol{S}_1(3\|1)$	0.1	0.12	0.11	0.09	0.10				
		(0.18)	(0.14)	(0.11)	(0.09)				
$\boldsymbol{S}_1(1\|2)$	0.25	0.29	0.27	0.28	0.28				
		(0.29)	(0.27)	(0.25)	(0.22)				
$\boldsymbol{S}_1(2\|2)$	0.6	0.41	0.42	0.45	0.45				
		(0.37)	(0.36)	(0.35)	(0.33)				
$\boldsymbol{S}_1(3\|2)$	0.15	0.31	0.31	0.27	0.27				
		(0.29)	(0.27)	(0.25)	(0.21)				

Note: GDP is under subjective beliefs: $\boldsymbol{S}_0 \neq \boldsymbol{T}_0$ and $\boldsymbol{S}_1 \neq \boldsymbol{T}_1$, but $\boldsymbol{S}_1(x_3) = \boldsymbol{T}_1(x_3)$.
Estimation with subjective beliefs is under the normalization $\boldsymbol{S}_1(x_3) = \boldsymbol{T}_1(x_3)$.

there is no need to estimate the additional predetermined continuation value for the terminal period. However, such an assumption is not appropriate in our application, because agents are still alive after the last decision period and the continuation value is likely to be nonzero after they stop working. To better describe agents' decisions, we allow the continuation value of the terminal period to be nonzero and specify the choice-specific value function in the terminal period as:

$$\tilde{u}(a, x, \epsilon) = \begin{cases} \epsilon_0, & \text{if } a = 0, \\ \tilde{u}(x) + \epsilon_1, & \text{if } a = 1. \end{cases}$$

The specification $u(x) = \tilde{u}(x)$ is equivalent to a zero continuation value for the terminal period (See Arcidiacono and Ellickson, 2011).

Note that our identification applies to discrete state variables, but income is continuous. Therefore, we divide the observed household income into three segments ($J = 3$):

$$x = \begin{cases} 1, & \text{if household income} \leq \$17,000, \\ 2, & \text{if } \$17,000 < \text{household income} \leq \$150,000, \\ 3, & \text{if household income} > \$150,000, \end{cases}$$

where $x = 1, 2, 3$ are referred to as low, medium, and high income, respectively. The cutoff of $17,000 is roughly the U.S. Department of Health & Human Services (HHS) poverty line for a family size of four, and $150,000 is the income level that leads to a "good" life in America according to a survey of WSL Strategic retail.[10]

10 Rosenberg, Yuval (2012, March 7), *The Fiscal Times*. Retrieved from https://www.businessinsider.com/the-basic-annual-income-every-american-would-need-is-150000-2012-3.

Table 3
Simulation results for a DGP of RE: finite horizon.

	True	Estimates with subjective beliefs				Estimates with rational expectations			
		$N = 300$	$N = 600$	$N = 1000$	$N = 2500$	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$
u_1	−2	−2.02	−2.02	−2.01	−2.01	−2.00	−2.00	−2.00	−2.00
		(0.21)	(0.15)	(0.11)	(0.07)	(0.14)	(0.10)	(0.08)	(0.05)
u_2	0.4	0.39	0.40	0.39	0.40	0.40	0.40	0.40	0.40
		(0.21)	(0.15)	(0.11)	(0.07)	(0.15)	(0.10)	(0.08)	(0.05)
u_3	2.1	2.14	2.13	2.12	2.11	2.10	2.10	2.10	2.10
		(0.36)	(0.24)	(0.18)	(0.12)	(0.20)	(0.14)	(0.11)	(0.07)
$\boldsymbol{S}_0(1\|1)$	0.8	0.76	0.76	0.76	0.77				
		(0.15)	(0.12)	(0.11)	(0.09)				
$\boldsymbol{S}_0(2\|1)$	0.1	0.17	0.16	0.16	0.15				
		(0.19)	(0.17)	(0.16)	(0.15)				
$\boldsymbol{S}_0(3\|1)$	0.1	0.07	0.07	0.07	0.08				
		(0.08)	(0.07)	(0.07)	(0.06)				
$\boldsymbol{S}_0(1\|2)$	0.2	0.27	0.25	0.26	0.24				
		(0.26)	(0.26)	(0.25)	(0.25)				
$\boldsymbol{S}_0(2\|2)$	0.6	0.48	0.51	0.50	0.52				
		(0.44)	(0.44)	(0.44)	(0.43)				
$\boldsymbol{S}_0(3\|2)$	0.2	0.25	0.24	0.24	0.23				
		(0.20)	(0.19)	(0.19)	(0.18)				
$\boldsymbol{S}_0(1\|3)$	0.1	0.11	0.11	0.10	0.10				
		(0.12)	(0.11)	(0.11)	(0.10)				
$\boldsymbol{S}_0(2\|3)$	0.19	0.20	0.18	0.19	0.19				
		(0.23)	(0.20)	(0.19)	(0.18)				
$\boldsymbol{S}_0(3\|3)$	0.71	0.70	0.71	0.71	0.71				
		(0.17)	(0.13)	(0.10)	(0.09)				

Note: GDP is rational expectation: $\boldsymbol{S}_0 = \boldsymbol{T}_0$ and $\boldsymbol{S}_1 = \boldsymbol{T}_1$.
Estimation allowing for subjective belief is with normalization of $\boldsymbol{S}_1$, i.e., $\boldsymbol{S}_1 = \boldsymbol{T}_1$.

Table 4
Simulation results for a DGP of SB: finite horizon.

	True	Estimates with subjective beliefs				Estimates with rational expectations			
		$N = 300$	$N = 600$	$N = 1000$	$N = 2500$	$N = 300$	$N = 600$	$N = 1000$	$N = 2500$
u_1	−2	−2.01	−2.01	−2.00	−2.01	−1.65	−1.65	−1.64	−1.64
		(0.19)	(0.14)	(0.11)	(0.07)	(0.13)	(0.10)	(0.07)	(0.04)
u_2	0.4	0.40	0.41	0.40	0.41	0.45	0.45	0.44	0.45
		(0.20)	(0.15)	(0.11)	(0.07)	(0.14)	(0.10)	(0.07)	(0.05)
u_3	2.1	2.14	2.13	2.12	2.11	1.72	1.72	1.72	1.72
		(0.33)	(0.23)	(0.18)	(0.11)	(0.17)	(0.12)	(0.09)	(0.06)
$\boldsymbol{S}_0(1\|1)$	0.9	0.88	0.88	0.89	0.89				
		(0.10)	(0.08)	(0.06)	(0.05)				
$\boldsymbol{S}_0(2\|1)$	0.05	0.09	0.08	0.08	0.07				
		(0.12)	(0.10)	(0.09)	(0.08)				
$\boldsymbol{S}_0(3\|1)$	0.05	0.04	0.04	0.04	0.04				
		(0.05)	(0.04)	(0.04)	(0.03)				
$\boldsymbol{S}_0(1\|2)$	0.1	0.25	0.23	0.22	0.19				
		(0.25)	(0.25)	(0.24)	(0.22)				
$\boldsymbol{S}_0(2\|2)$	0.8	0.53	0.58	0.59	0.63				
		(0.43)	(0.42)	(0.42)	(0.39)				
$\boldsymbol{S}_0(3\|2)$	0.1	0.22	0.20	0.19	0.17				
		(0.19)	(0.19)	(0.19)	(0.17)				
$\boldsymbol{S}_0(1\|3)$	0.05	0.05	0.05	0.05	0.05				
		(0.07)	(0.06)	(0.06)	(0.05)				
$\boldsymbol{S}_0(2\|3)$	0.095	0.09	0.09	0.09	0.09				
		(0.13)	(0.11)	(0.10)	(0.09)				
$\boldsymbol{S}_0(3\|3)$	0.855	0.85	0.86	0.85	0.86				
		(0.11)	(0.08)	(0.06)	(0.05)				

Note: GDP is under the subjective beliefs: $\boldsymbol{S}_0 \neq \boldsymbol{T}_0$ but $\boldsymbol{S}_1 = \boldsymbol{T}_1$.
Estimation with subjective beliefs is under the normalization $\boldsymbol{S}_1 = \boldsymbol{T}_1$.

We impose two assumptions to identify the model. First, we assume that subjective beliefs are homogeneous, even though women in our sample differ in age and education.[11] We believe that homogeneity of subjective beliefs is a

[11] We can easily incorporate heterogeneous subjective beliefs in our analysis. However, such an approach would require a larger sample size.

Table 5
Simulation results for a DGP of RE: infinite horizon.

	True	Estimates with subjective beliefs				Estimates with rational expectations			
		$N=300$	$N=600$	$N=1000$	$N=2500$	$N=300$	$N=600$	$N=1000$	$N=2500$
$u_1(x_1)$	0.1	0.08	0.09	0.09	0.09	0.10	0.11	0.10	0.10
		(0.16)	(0.11)	(0.08)	(0.06)	(0.17)	(0.12)	(0.09)	(0.06)
$u_1(x_2)$	0.2	0.21	0.21	0.21	0.20	0.20	0.20	0.20	0.20
		(0.18)	(0.12)	(0.10)	(0.07)	(0.18)	(0.13)	(0.10)	(0.06)
$u_2(x_1)$	0.2	0.16	0.16	0.17	0.18	0.20	0.20	0.20	0.20
		(0.09)	(0.08)	(0.07)	(0.06)	(0.06)	(0.05)	(0.03)	(0.02)
$u_2(x_2)$	−0.2	−0.18	−0.17	−0.19	−0.20	−0.20	−0.20	−0.20	−0.20
		(0.10)	(0.08)	(0.08)	(0.07)	(0.07)	(0.05)	(0.04)	(0.02)
$\boldsymbol{S}_0(1\|1)$	0.6	0.84	0.81	0.77	0.71				
		(0.33)	(0.33)	(0.36)	(0.36)				
$\boldsymbol{S}_0(2\|1)$	0.45	0.30	0.29	0.37	0.44				
		(0.44)	(0.42)	(0.44)	(0.43)				

Note: GDP is under rational expectations: $\boldsymbol{S}_0^x = \boldsymbol{T}_0^x$ and $\boldsymbol{S}_1^x = \boldsymbol{T}_1^x$.
Estimation with subjective beliefs is under the normalization $\boldsymbol{S}_1^x = \boldsymbol{T}_1^x$.

Table 6
Simulation results for a DGP of SB: infinite horizon.

	True	Estimates with subjective beliefs				Estimates with rational expectations			
		$N=300$	$N=600$	$N=1000$	$N=2500$	$N=300$	$N=600$	$N=1000$	$N=2500$
$u_1(x_1)$	0.1	0.09	0.09	0.09	0.10	0.11	0.11	0.11	0.11
		(0.16)	(0.11)	(0.09)	(0.06)	(0.17)	(0.12)	(0.09)	(0.06)
$u_1(x_2)$	0.2	0.21	0.19	0.19	0.19	0.19	0.18	0.19	0.19
		(0.17)	(0.13)	(0.10)	(0.07)	(0.17)	(0.13)	(0.10)	(0.06)
$u_2(x_1)$	0.2	0.18	0.19	0.19	0.19	0.22	0.22	0.22	0.22
		(0.09)	(0.08)	(0.07)	(0.06)	(0.06)	(0.04)	(0.03)	(0.02)
$u_2(x_2)$	−0.2	−0.20	−0.20	−0.21	−0.21	−0.23	−0.22	−0.23	−0.23
		(0.10)	(0.09)	(0.09)	(0.08)	(0.06)	(0.05)	(0.04)	(0.02)
$\boldsymbol{S}_0(1\|1)$	0.7	0.81	0.77	0.76	0.72				
		(0.33)	(0.35)	(0.34)	(0.31)				
$\boldsymbol{S}_0(2\|1)$	0.3	0.28	0.33	0.36	0.35				
		(0.42)	(0.43)	(0.43)	(0.40)				

Note: GDP is under subjective beliefs: $\boldsymbol{S}_0^x \neq \boldsymbol{T}_0^x$ but $\boldsymbol{S}_1^x = \boldsymbol{T}_1^x$.
Estimation with subjective beliefs is under the normalization $\boldsymbol{S}_1^x = \boldsymbol{T}_1^x$.

Table 7
Distribution of observations (by number of years).

# years	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
# obs	101	109	135	105	102	104	102	94	89	94	79	91	85	85	80	71	147
cum %	6.04	12.55	20.6	26.9	33.0	39.2	45.3	50.9	56.3	61.9	66.6	72.0	77.1	82.2	87.0	91.2	100

Table 8
Descriptive statistics.

	# of observations	Mean	Std. Dev.	5-th pctile	median	95-th pctile
Age	22, 941	48.84	5.96	40	49	59
Education†	22, 941	3.98	1.84	0	4	7
Annual income (10K $)	22, 941	5.77	6.02	0.67	4.81	13.31
Employment	22, 941	.58	.49	0	1	1

Note: The number of observations is aggregated at individual-by-year level. Income is in 1999 dollars. Education is classified into nine groups. 1: 0–5 grades; 2: 6–8 grades; 3: some high school; 4: completed high school; 5: 12 grades plus non-academic training; 6: college, no degree; 7: college, bachelors degree; 8: college, advanced or professional degree, some graduate work; 9: not reported.

reasonable first-order approximation because we only focus on women between ages 39 to 60. Arguably, they are sufficiently experienced that age and eduction would not significantly affect their subjective beliefs. Second, we assume that subjective beliefs about the future income distribution for those high-income households with a working wife, i.e., $s(x'|x = 3, a = 1)$, is known to be the same as the objective transition observed in the data. This normalization is required for identification as stated in Assumption 6A. We impose this restriction because the future income for a high-income household with a working wife is less uncertain than in the other cases. For example, high-income households

Table 9
Estimates of subjective, objective beliefs and preference parameters.

		Transition ($a=0$)			Transition ($a=1$)			Preference	
		Low	Medium	High	Low	Medium	High	Stationary	Ending
sub.	Low	1.000	0.000	0.000	0.748	0.249	0.003	−0.611	−0.686
		(0.000)	(0.000)	(0.000)	(0.169)	(0.181)	(0.075)	(0.238)	(0.160)
	Medium	0.150	0.850	0.000	0.000	1.000	0.000	0.224	0.111
		(0.058)	(0.186)	(0.156)	(0.000)	(0.000)	(0.000)	(0.049)	(0.112)
	High	0.000	0.999	0.001	–	–	–	0.335	0.132
		(0.049)	(0.186)	(0.168)	–	–	–	(0.169)	(0.580)
obj.	Low	0.749	0.249	0.002	0.754	0.244	0.001	−0.243	−0.748
		(0.015)	(0.015)	(0.001)	(0.017)	(0.017)	(0.001)	(0.086)	(0.146)
	Medium	0.069	0.921	0.011	0.039	0.947	0.015	0.413	0.149
		(0.004)	(0.004)	(0.002)	(0.002)	(0.003)	(0.002)	(0.041)	(0.099)
	High	0.007	0.237	0.756	0.002	0.294	0.704	0.496	0.272
		(0.005)	(0.037)	(0.037)	(0.002)	(0.032)	(0.033)	(0.200)	(0.472)

Note: The columns "ending" and "stationary" are corresponding to the estimates of flow utility in the terminal period and other periods, respectively.

may not qualify for some social welfare programs as low-income households do, so it is relatively easier for high-income households to predict their future income if the wife works.

7.3. Estimation results

To estimate the model, we use the setting described in Theorem 1, where identification requires at least $2J = 6$ periods of observations. Thus, for our sample we retain those who appear in at least six periods.

Table 9 presents the estimation results of both subjective beliefs and rational expectations, and the preferences.[12] The top panel provides estimates of transition matrices and parameters of utility under subjective beliefs. For comparison, the bottom panel displays estimates obtained by imposing the assumption of rational expectations, for which the parameters are estimated by using maximum likelihood directly on the data. Table 9 reveals discrepancies between subjective beliefs versus rational expectations. We formally test whether the two sets of transition matrices are the same—i.e., whether households hold rational expectations about income transitions—by using a Wald test. Conditional on both non-working and working status, we reject at the 1% significance level the null hypothesis that subjective beliefs and their objective counterparts are equal. The results indicate that agents do not have perfect foresight about income transition. This implies that our understanding of agents' working decisions under the assumption of rational expectations could be misleading. We also use the Wald test to determine whether households' subjective beliefs vary conditional on the wife's working status. We find that at the 1% significance level subjective beliefs of households with a working wife differ significantly from those with a non-working wife. We further conduct a similar Wald test for the objective transitions of income and obtain similar results. These results demonstrate that agents are sophisticated enough to predict the different impacts of their actions on income transitions, even though they do not have rational expectations.

Next we test whether households' subjective beliefs are stationary or not over time. The women in our sample range from 39 to 60, so there are 21 periods (years), except for the terminal period. Because identification requires at least 6 periods of observations, we divide the whole sample period into three equal stages: age 39 to 45; age 46 to 52; and age 53 to 59. Let θ_{1s}, θ_{2s}, and θ_{3s} denote the parameters of subjective beliefs in the three stages, respectively. We assume that households are not aware of the future change in beliefs when they make decisions, and that their preferences do not change over time.

Note that normalization is required for each set of subjective beliefs. We use the objective transition in the same state to pin down the subjective beliefs for high-income households with a working wife, i.e., $s(x'|x = 3, a = 1)$. We jointly estimate the flow utility, continuation value in the ending period, and the three different sets of subjective beliefs by maximizing the likelihood function. Let $\theta \equiv (\theta_u, \theta_{1s}^e, \theta_{2s}^e, \theta_{3s}^e)$ be the parameters of preferences and subjective beliefs, the log-likelihood function is

$$Q_n(\theta) = \max_{\theta} \sum_{i=1}^{n} \left(\sum_{t=53}^{60} \log p_t(a_{it}|x_{it}; \theta_u, \theta_{3s}^e, \hat{\theta}_s^n) + \sum_{t=46}^{52} \log p_t(a_{it}|x_{it}; \theta_u, \theta_{2s}^e, \hat{\theta}_s^n) \right.$$
$$\left. + \sum_{t=39}^{45} \log p_t(a_{it}|x_{it}; \theta_u, \theta_{1s}^e, \hat{\theta}_s^n) \right), \tag{32}$$

12 Before estimating the model parameters, we test the rank of the observed matrix $\Delta\xi_{i,K}$ and find that it is full rank, i.e., Assumption 5A holds.

where θ_s^n is obtained from the estimated objective state transition by using all of the data. Based on the estimates of $\hat{\theta}$, we can use the Likelihood-Ratio (LR) test to check the stationarity of agents' subjective beliefs, where the null hypothesis is

$$H_0 : \theta_{1s}^e = \theta_{2s}^e = \theta_{3s}^e.$$

The LR test statistic is

$$LR = 2 \cdot (Q_n(\hat{\theta}) - Q_n(\tilde{\theta})), \tag{33}$$

where $\hat{\theta}$ and $\tilde{\theta}$ are the MLE estimates of θ under the alternative and null hypothesis, respectively. Because $\theta_{js}^e, j = 1, 2, 3$ contain 10 independent parameters, the LR test statistic is asymptotically distributed according to a chi-squared distribution with a degree of freedom of 20. We find that the p-value of the test is 0.084. Thus we fail to reject the null hypothesis that subjective beliefs are stationary at a significance level of 5%. However, we stress that these findings are based on the *ad hoc* assumption that beliefs are unchanged for each phase and women are unaware of the change in their subjective beliefs in the future. While these findings are useful for us to justify Assumption 2(b), one should be cautious in using the results as evidence against learning.

There are two important observations from the estimated transition matrices in Table 9. First, households with a non-working wife are overly pessimistic about their income transitions; those with a working wife are less so. For example, among medium-income households with a non-working wife, agents expect their household income to stay in the medium category with probability 0.85 and to drop into the low income category with probability 0.15. By contrast, the objective transition probabilities for income staying at medium, dropping to low, or increasing to high are 0.92, 0.07, and 0.01, respectively. A similar pattern can be seen for households of low and high income. On the other hand, for households with a working wife, subjective beliefs are very close to the objective transition probabilities, even though rational expectations are rejected as we described above. This finding demonstrates that deviation of subjective beliefs from rational expectations is action dependent. Since investigation of expectation formation is beyond the scope of this paper, we leave this to future work.

Second, agents have "asymmetric" beliefs about their income transitions. In households with a non-working wife, agents of medium-income households believe that income will remain at medium with probability 0.85, which is about 0.07 below the corresponding objective probability. However, agents of high-income households are more pessimistic: they believe with almost certainty that income will drop to medium, while the objective probability of this drop is just 0.24. This finding is consistent with survey data suggesting asymmetric beliefs of agents. For example, Heimer et al. (2019) find discrepancies between surveyed mortality expectations and actuarial statistics from the Social Security administration, and these discrepancies differ across age groups.

Under both subjective beliefs and rational expectations, estimated preferences share a similar pattern: women prefer to work if their household income is medium or high; they prefer not to work if their current income is low. This may be because women in low-income households likely have less educational attainment, and consequently face less attractive job options. This may explain a reluctance to work. Our estimates of agent preferences indicate that the utility of the terminal period is different from the flow utility under both rational and subjective beliefs. This implies that the continuation value after the terminal period is not zero, at least for this dataset.

Next, we investigate how the discrepancies between subjective and rational expectations affect women's labor force participation. For this purpose, we conduct a counterfactual analysis, using the estimates in Table 9 to simulate CCPs and to compute percentage differences under subjective and rational expectations. The results presented in Table 10 suggest: (1) having rational expectations would decrease labor participation; and (2) the impact of subjective beliefs on labor participation choice is heterogeneous. Regardless of household income level, women with rational expectations would be less likely to work. However, this disparity decreases as the women approach age 60. The difference in CCPs between the two sets of beliefs for women in low-income households is about three times that of those in the medium and high-income categories. For example, at age 57, the percentage difference of CCPs for low, medium, and high-income categories are 12.8%, 4.4%, and 3.7%, respectively. This counterfactual analysis has important policy implications. If a policymaker aims to promote labor participation of women, then providing them with accurate information about income transitions would not work, because there is already an oversupply of labor due to their subjective beliefs.

8. Concluding remarks

This paper studies DDC models with agents holding subjective beliefs about the state transition; these beliefs may be different from the objective transition probabilities observed in the data. We show that agents' subjective beliefs and preferences are nonparametrically identified in both finite and infinite horizon cases. The identification in both the infinite and finite horizon frameworks relies on exclusion restrictions. We propose estimating the model using a maximum likelihood estimator, and we present Monte Carlo evidence to illustrate that our estimator performs well with mid-sized samples. Applying our methodology to PSID data, we illustrate that households do not hold rational expectations about their income transitions. The discrepancies between their subjective beliefs and the objective transition probabilities may lead to a significant difference in women's labor force participation. Our estimates also shed light on how subjective beliefs

Table 10
Simulated conditional choice probabilities.

	Sub. belief.			Rational exp.			Percentage diff.		
	$x = 1$	$x = 2$	$x = 3$	$x = 1$	$x = 2$	$x = 3$	$x = 1$	$x = 2$	$x = 3$
$t = 55$	0.423	0.599	0.614	0.351	0.563	0.581	−17.1%	−6.1%	−5.3%
$t = 56$	0.414	0.594	0.610	0.351	0.562	0.581	−15.2%	−5.3%	−4.6%
$t = 57$	0.402	0.587	0.604	0.351	0.561	0.582	−12.8%	−4.4%	−3.7%
$t = 58$	0.388	0.579	0.597	0.351	0.560	0.582	−9.6%	−3.2%	−2.4%
$t = 59$	0.371	0.568	0.584	0.351	0.558	0.583	−5.2%	−1.7%	−0.2%
$t = 60$	0.335	0.528	0.533	0.335	0.528	0.533	0.0%	0.0%	0.0%

Note: The percentage difference is defined as $\left[\text{CCPs (sub. belief.)} - \text{CCPs (rational. exp.)}\right]/\text{CCPs (sub. belief.)}$.

affect agents' dynamic decisions and what policies would be appropriate for improving labor market participation among women.

A direction for future research is to relax some important assumptions in this paper, e.g., invariant subjective beliefs, and to incorporate learning into the model. While our method is introduced in the context of discrete choice, it may be possible to extend it to dynamic models with continuous choice, e.g., life-cycle consumption problems. We are considering these possibilities for future work.

Appendix. Proofs

This section provides proofs for all the identification results.

A.1. Proof of Theorem 1

The proof of Theorem 1 is sketched in the main text. Therefore, we only provide proofs for the main steps used to derive Theorem 1.

Derivation of Eq. (4). To derive the matrix representation of the log ratio of CCPs, we first rewrite the summation of the value function in a matrix representation, taking into account the construction of the belief vector. That is,

$$\begin{aligned}\sum_{x'=1}^{J} V_{t+1}(x')s(x'|x,a) &= \sum_{x'=1}^{J-1} V_{t+1}(x')s(x'|x,a) + \beta V_{t+1}(J)\Big[1 - \sum_{x'=1}^{J-1} s(x'|x,a)\Big] \\ &= \sum_{x'=1}^{J-1}\big[V_{t+1}(x') - V_{t+1}(J)\big]s(x'|x,a) + \beta V_{t+1}(J). \\ &\equiv S_a(x)\boldsymbol{V}_{t+1} + \beta V_{t+1}(J). \end{aligned} \tag{A.1}$$

Consequently, the log ratio of CCPs for any t can be represented as:

$$\begin{aligned}\xi_{t,i,K}(x) \equiv \log\left(\frac{p_{t,i}(x)}{p_{t,K}(x)}\right) &= \big[u_i(x) - u_K(x)\big] + \beta\sum_{x'} V_{t+1}(x')\big[s(x'|x,i) - s(x'|x,K)\big] \\ &= u_i(x) - u_K(x) + \beta\big[S_i(x) - S_K(x)\big]\boldsymbol{V}_{t+1}.\end{aligned} \tag{A.2}$$

Derivation of Eq. (7). We derive the recursive relationship of value functions over time. We first express the *ex-ante* value function using the choice-specific value function $v_{t,K}(x)$ with an adjustment of $-\log p_{t,K}(x)$. Specifically, the *ex-ante* value function at t can be expressed as

$$\begin{aligned}V_t(x) &= -\log p_{t,K}(x) + v_{t,K}(x) \\ &= -\log p_{t,K}(x) + u_K(x) + \beta\sum_{x'} V_{t+1}(x')s(x'|x,a) \\ &= -\log p_{t,K}(x) + u_K(x) + \beta S_K(x)\boldsymbol{V}_{t+1} + \beta V_{t+1}(J).\end{aligned}$$

Consequently, the *ex-ante* value function for state $x \in \mathcal{X}, x \neq J$ relative to J in period t can be represented as

$$\begin{aligned}V_t(x) - V_t(J) &= \big[-\log p_{t,K}(x) + u_K(x) + \beta S_K(x)\boldsymbol{V}_{t+1} + \beta V_{t+1}(J)\big] \\ &\quad - \big[-\log p_{t,K}(J) + u_K(J) + \beta S_K(J)\boldsymbol{V}_{t+1} + \beta V_{t+1}(J)\big] \\ &= -\big[\log p_{t,K}(x) - \log p_{t,K}(J)\big] + \big[u_K(x) - u_K(J)\big] + \beta\big[S_K(x) - S_K(J)\big]\boldsymbol{V}_{t+1}.\end{aligned}$$

We stack the equation above for all state $x, x \neq J$ and obtain the following matrix form.

$$\begin{aligned}
\boldsymbol{V}_t &\equiv \begin{pmatrix} V_t(1)-V_t(J) \\ V_t(2)-V_t(J) \\ \dots \\ V_t(J-1)-V_t(J) \end{pmatrix} \\
&= -\begin{pmatrix} \log \boldsymbol{p}_{t,K}(1)-\log \boldsymbol{p}_{t,K}(J) \\ \log \boldsymbol{p}_{t,K}(2)-\log \boldsymbol{p}_{t,K}(J) \\ \dots \\ \log \boldsymbol{p}_{t,K}(J-1)-\log \boldsymbol{p}_{t,K}(J) \end{pmatrix} + \begin{pmatrix} u_K(1)-u_K(J) \\ u_K(2)-u_K(J) \\ \dots \\ u_K(J-1)-u_K(J) \end{pmatrix} + \beta \begin{pmatrix} S_K(1)-S_K(J) \\ S_K(2)-S_K(J) \\ \dots \\ S_K(J-1)-S_K(J) \end{pmatrix} \boldsymbol{V}_{t+1} \\
&\equiv -\log \boldsymbol{p}_{t,K} + \boldsymbol{u}_K + \widetilde{\boldsymbol{S}}_K \boldsymbol{V}_{t+1},
\end{aligned}$$

where $\log \boldsymbol{p}_{t,K}$ and $\boldsymbol{u}_K$ are defined similarly to $\boldsymbol{V}_t$, and $\widetilde{\boldsymbol{S}}_K \equiv \left[S_K(1)-S_K(J), S_K(2)-S_K(J), \dots, S_K(J-1)-S_K(J)\right]'$. Consequently, we have the following recursive relationship of *ex-ante* value functions over time.

$$\boldsymbol{V}_t = -\log \boldsymbol{p}_{t,K} + \boldsymbol{u}_K + \widetilde{\boldsymbol{S}}_K \boldsymbol{V}_{t+1}. \tag{A.3}$$

Identification of beliefs associated with other actions. In what follows, we show that the subjective beliefs associated with other actions i', $i' \neq i$ and $i' \neq K$, can be identified. First we show that the *ex ante* value functions $\Delta \boldsymbol{V}_{t+1}$ can be identified for period $t = 2, \dots, T-1$ through Eq. (6) with the belief matrices $\boldsymbol{S}_i$ and $\boldsymbol{S}_K$ being identified.

$$\Delta \boldsymbol{\xi}_{t,i,K} = \beta[\boldsymbol{S}_i - \boldsymbol{S}_K]\Delta \boldsymbol{V}_{t+1}. \tag{A.4}$$

We augment this equation into a matrix form:

$$\Delta \boldsymbol{\xi}^2_{i,K} = \beta[\boldsymbol{S}_i - \boldsymbol{S}_K]\Delta \boldsymbol{V}^2, \tag{A.5}$$

Note that $\Delta \boldsymbol{\xi}^2_{i,K} = \begin{bmatrix} \Delta \boldsymbol{\xi}_{2,i,K} & \Delta \boldsymbol{\xi}_{3,i,K} & \dots & \Delta \boldsymbol{\xi}_{T-1,i,K} \end{bmatrix}$ and $\Delta \boldsymbol{V}^2 \equiv \begin{bmatrix} \Delta \boldsymbol{V}_2, & \Delta \boldsymbol{V}_3, & \dots & \Delta \boldsymbol{V}_{T-1} \end{bmatrix}$. The matrix $\Delta \boldsymbol{\xi}^2_{i,K}$, size of $(J-1)\times(T-2)$, is full row rank because it is one part of the matrix $\Delta \boldsymbol{\xi}_{i,K}$, size of $2(J-1)\times(T-2)$, is full row rank. The full rank Assumption 4 allows us to identify $\Delta \boldsymbol{V}^2$ as $\beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\Delta \boldsymbol{\xi}^2_{i,K}$.

We then can identify the beliefs for action i', $i' \neq i$ and $i' \neq K$ as follows.

$$\begin{aligned}
\Delta \boldsymbol{\xi}^2_{i',K} &= \beta[\boldsymbol{S}_{i'} - \boldsymbol{S}_K]\Delta \boldsymbol{V}^2 \\
&= \beta[\boldsymbol{S}_{i'} - \boldsymbol{S}_K]\beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\Delta \boldsymbol{\xi}^2_{i,K} \\
\leftrightarrow \quad \boldsymbol{S}_{i'} - \boldsymbol{S}_K &= \Delta \boldsymbol{\xi}^2_{i',K}[\Delta \boldsymbol{\xi}^2_{i,K}]^{+}[\boldsymbol{S}_i - \boldsymbol{S}_K] \\
\leftrightarrow \quad \boldsymbol{S}_{i'} &= \boldsymbol{S}_K + \Delta \boldsymbol{\xi}^2_{i',K}[\Delta \boldsymbol{\xi}^2_{i,K}]^{+}[\boldsymbol{S}_i - \boldsymbol{S}_K]
\end{aligned} \tag{A.6}$$

The first equality holds due to equation eq: identify i' for all other choices $i' \neq i, K$. The second equality plugging in the identified $\Delta \boldsymbol{V}^2$. The third equality holds with $[\Delta \boldsymbol{\xi}^2_{i,K}]^{+}$ being the right inverse of the matrix $\Delta \boldsymbol{\xi}^2_{i,K}$.

A.2. Proof of Theorem 2

First we stack all moment conditions involving the log ratios of CCPs in Eq. (10) in the following matrix representation:

$$\widetilde{\boldsymbol{S}}_K[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\Delta \boldsymbol{\xi}^1_{i,K} - \beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\Delta \boldsymbol{\xi}^2_{i,K} = \Delta \log \boldsymbol{p}_K.$$

When the belief matrix associated with action K is known, we can rewrite the equation above in the following vectorization expression:

$$\begin{aligned}
vec(\Delta \log \boldsymbol{p}_K) &= vec(\widetilde{\boldsymbol{S}}_K[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\Delta \boldsymbol{\xi}^1_{i,K}) - vec(\beta^{-1}[\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}\Delta \boldsymbol{\xi}^2_{i,K}) \\
&= [(\Delta \boldsymbol{\xi}^1_{i,K})' \otimes (\widetilde{\boldsymbol{S}}_K)]vec([\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}) - \beta^{-1}[(\Delta \boldsymbol{\xi}^2_{i,K})' \otimes I]vec([\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}) \\
&= [(\Delta \boldsymbol{\xi}^1_{i,K})' \otimes (\widetilde{\boldsymbol{S}}_K) - \beta^{-1}(\Delta \boldsymbol{\xi}^2_{i,K})' \otimes I]vec([\boldsymbol{S}_i - \boldsymbol{S}_K]^{-1}),
\end{aligned} \tag{A.7}$$

where $[(\Delta \boldsymbol{\xi}^1_{i,K})' \otimes (\widetilde{\boldsymbol{S}}_K) - \beta^{-1}(\Delta \boldsymbol{\xi}^2_{i,K})' \otimes I]$ is a $(T-2)\cdot(J-1)$ by $(J-1)\cdot(J-1)$ matrix. Identification requires $[(\Delta \boldsymbol{\xi}^1_{i,K})' \otimes (\beta\widetilde{\boldsymbol{S}}_K) - (\Delta \boldsymbol{\xi}^2_{i,K})' \otimes I]$ is of full column rank, which implicitly imposes the restriction $T-2 \geq J-1$. Again, the full rank condition is empirically testable.

A.3. Identification with the terminal period

In this subsection, we show that the model can be identified using fewer periods of data than the one required in Theorems 1–2 if data on the terminal period are available, i.e., T is the terminal period.

If we assume that the continuation value in the terminal period is zero, agents do not need to form beliefs for the future at the terminal period. Thus, CCPs in the terminal period allow us to identify the relative preference $[u_i(x) - u_K(x)]$, regardless agents have subjective or rational expectations. Once the flow utility is identified by using CCPs in the terminal period, the impact of preference on the log ratio of CCPs is known. Consequently, we do not need to eliminate the utility to identify subjective beliefs as in Theorems 1–2. Specifically, we rewrite Eq. (4) as:

$$\begin{aligned}\eta_{t,i,K}(x) &\equiv \xi_{t,i,K}(x) - [u_i(x) - u_K(x)]\\ &= \beta[S_i(x) - S_K(x)]\boldsymbol{V}_{t+1},\end{aligned} \tag{A.8}$$

where $\eta_{t,i,K}(x)$ is identified because the utility difference $u_i(x) - u_K(x)$ is identified. We collect all variations of $\eta_{t,i,K}(x)$ over time in the following matrix:

$$\boldsymbol{\eta}_{i,K} \equiv \begin{bmatrix} \boldsymbol{\eta}_{2,i,K} & \boldsymbol{\eta}_{4,i,K} & \cdots & \boldsymbol{\eta}_{T,i,K} \\ \boldsymbol{\eta}_{1,i,K} & \boldsymbol{\eta}_{3,i,K} & \cdots & \boldsymbol{\eta}_{T-1,i,K} \end{bmatrix},$$

where $\boldsymbol{\eta}_{t,i,K}$ is defined analogously to the matrix collecting all log ratio of CCPs $\Delta\boldsymbol{\xi}_{t,i,K}$. Similar to Theorems 1–2, we can identify the subjective beliefs by imposing a rank condition stated in the following:

Assumption A.3.1. (a) The number of periods observed is not smaller than $2J-1$, i.e, $T \geq 2J-1$. (b) The matrix $\boldsymbol{\eta}_{i,K}$ is of full row rank.

Similar to Assumptions 5A and 5B, Assumption A.3.1 is also testable. We summarize the identification result in the following corollary to Theorem 1.

Corollary 1. *Suppose that Assumptions 1–4, A.3.1, and 6A hold. Then the subjective beliefs $s(x'|x,a)$ for $x, x' \in \{1, 2, \ldots, J\}$ and $a \in \{1, 2, \ldots, K\}$ are identified as a closed-form function of the CCPs, $p_t(a|x)$, for $t = 1, \ldots, T$, where $T \geq 2J-1$.*

Corollary 1 shows that $2J-1$ periods of data (versus $2J$ periods required in Theorem 1) are sufficient for identification if the terminal period of data are available.

Analogously, if the terminal period is observed and Assumption 6B is imposed, i.e., $\boldsymbol{S}_K$ is known, we can improve upon Theorem 2 by identifying the model with $J-1$ periods of data. We provide some brief discussions on the identification as the procedure is similar to that of Corollary 1 . First of all, utility function can be recovered from the choice in the terminal period. Using this information and the known subjective beliefs $\boldsymbol{S}_K$, we can identify value function $\boldsymbol{V}_t$ for $t = 1, 2, \ldots, T$. To identify the beliefs $\boldsymbol{S}_i$, we only need to use Eq. (A.8) with $\boldsymbol{S}_i$ being the only unknown. We define

$$\widetilde{\boldsymbol{\eta}}_{i,K} \equiv \left[\boldsymbol{\eta}_{1,i,K}, \boldsymbol{\eta}_{2,i,K}, \ldots, \boldsymbol{\eta}_{T,i,K}\right],$$

which is an observed $(J-1) \times T$ matrix. A testable full rank condition is necessary for identification.

Assumption A.3.2. (a) The number of periods observed is not smaller than $J-1$, i.e., $T \geq J-1$. (b) The matrix $\widetilde{\boldsymbol{\eta}}_{i,K}$ is of full row rank.

We present the identification result under Assumption A.3.2 as a corollary to Theorem 2:

Corollary 2. *Suppose that Assumptions 1–4, A.3.2, and 6B hold. Then the subjective beliefs $s(x'|x,a)$ for $x, x' \in \{1, 2, \ldots, J\}$ and $a \in \{1, 2, \ldots, K\}$, are identified as a closed-form function of the CCPs, $p_t(a|x)$ for $t = 1, \ldots, T$, where $T \geq J-1$*

A.4. Proof of Theorem 3

This section provides all necessary proofs for Theorem 3.

Derivation of Eq. (13). To derive the matrix representation of the log ratio of CCPs, we first rewrite the summation of the value function in a matrix representation, taking into account the construction of the belief vector. That is,

$$\begin{aligned}&\sum_{x'=1}^{J}\sum_{w'=1}^{M} V_{t+1}(x', w')s(x'|x,a)f(w'|w)\\ &= \sum_{x'=1}^{J-1}\sum_{w'=1}^{M} V_{t+1}(x', w')s(x'|x,a)f(w'|w) + \beta\sum_{w'=1}^{M} V_{t+1}(J, w')f(w'|w)\Big[1 - \sum_{x'=1}^{J-1} s(x'|x,a)\Big]\\ &= \sum_{x'=1}^{J-1}\sum_{w'=1}^{M}\Big[V_{t+1}(x', w') - V_{t+1}(J, w')\Big]s(x'|x,a)f(w'|w) + \beta\sum_{w'=1}^{M} V_{t+1}(J, w')f(w'|w).\\ &\equiv \sum_{w'=1}^{M} S_a(x)V_{t+1}(w')f(w'|w) + \beta\sum_{w'=1}^{M} V_{t+1}(J, w')f(w'|w)\\ &\equiv S_a(x)\boldsymbol{V}_{t+1}F(w) + \overline{V}_{t+1,w},\end{aligned} \tag{A.9}$$

where

$$\begin{aligned}
V_{t+1}(w) &\equiv [V_{t+1}(x=1,w)-V_{t+1}(J,w),\ldots,V_{t+1}(x=J-1,w)-V_{t+1}(J,w)]',\\
F(w) &= [f(w'=1|w),\ldots,f(w'=M|w)]',\\
\boldsymbol{V}_{t+1} &\equiv [V_{t+1}(w=1),\ldots,V_{t+1}(w=M)],\\
\overline{V}_{t+1,w} &\equiv \beta\sum_{w'=1}^{M} V_{t+1}(J,w')f(w'|w).
\end{aligned}$$

Consequently, the log ratio of CCPs in period t can be represented as:

$$\begin{aligned}
\xi_{t,i,K}(x,w) &\equiv \log\left(\frac{p_{t,i}(x,w)}{p_{t,K}(x,w)}\right)\\
&= \left[u_i(x,w)-u_K(x,w)\right]+\beta\sum_{x'}\sum_{w'=1}^{M}V_{t+1}(x')\left[s(x'|x,i)-s(x'|x,K)\right]f(w'|w)\\
&= \left[u_i(x,w)-u_K(x,w)\right]+\beta\left[S_i(x)-S_K(x)\right]\boldsymbol{V}_{t+1}F(w).
\end{aligned} \tag{A.10}$$

Derivation of Eq. (16). Under the assumption that $\boldsymbol{S}_i-\boldsymbol{S}_K$ is invertible, we consider (15) for both t and $t+1$,

$$\begin{aligned}
\Delta\boldsymbol{V}_{t+1}\boldsymbol{F}_w &= \beta^{-1}(\boldsymbol{S}_i-\boldsymbol{S}_K)^{-1}\Delta\boldsymbol{\xi}_{t,i,K},\\
\Delta\boldsymbol{V}_{t}\boldsymbol{F}_w &= \beta^{-1}(\boldsymbol{S}_i-\boldsymbol{S}_K)^{-1}\Delta\boldsymbol{\xi}_{t-1,i,K}.
\end{aligned} \tag{A.11}$$

We then follow similar argument to the case without the additional state variable w to derive the recursive relationship of the first differences of *ex-ante* value functions. First of all, we can represent the *ex-ante* value function using the choice-specific value function $v_{t,K}(x,w)$ with an adjustment of the corresponding CCPs, i.e., $-\log p_{t,K}(x,w)$.

$$\begin{aligned}
V_t(x,w) &= -\log p_{t,K}(x,w)+v_{t,K}(x,w)\\
&= -\log p_{t,K}(x,w)+u_K(x,w)+\beta\sum_{x'=1}^{J}\sum_{w'=1}^{M}V_{t+1}(x',w')s(x'|x,K)f(w'|w)\\
&= -\log p_{t,K}(x,w)+u_K(x,w)+\beta S_K(x)\boldsymbol{V}_{t+1}F(w)+\overline{V}_{t+1,w}.
\end{aligned}$$

The *ex-ante* value relative to the reference state $x=J$ for any w is

$$\begin{aligned}
&V_t(x,w)-V_t(J,w)\\
&= \left[-\log p_{t,K}(x,w)+u_K(x,w)+\beta S_K(x)\boldsymbol{V}_{t+1}F(w)+\overline{V}_{t+1,w}\right]\\
&- \left[-\log p_{t,K}(J,w)+u_K(J,w)+\beta S_K(J)\boldsymbol{V}_{t+1}F(w)+\overline{V}_{t+1,w}\right]\\
&= -\left[\log p_{t,K}(x,w)-\log p_{t,K}(J,w)\right]+\left[u_K(x,w)-u_K(J,w)\right]+\beta\left[S_K(x)-S_K(J)\right]\boldsymbol{V}_{t+1}F(w)
\end{aligned}$$

By the construction of $\boldsymbol{V}_t$, we stack $V_t(x,w)-V_t(J,w)$ for $x\in\{1,2,\ldots,J-1\}$ and all w,

$$\begin{aligned}
\boldsymbol{V}_t &= \begin{pmatrix}
V_t(1,1)-V_t(J,1) & V_t(1,2)-V_t(J,2) & \ldots & V_t(1,M)-V_t(J,M)\\
V_t(2,1)-V_t(J,1) & V_t(2,2)-V_t(J,2) & \ldots & V_t(2,M)-V_t(J,M)\\
\ldots & & & \\
V_t(J-1,1)-V_t(J,1) & V_t(J-1,2)-V_t(J,2) & \ldots & V_t(J-1,M)-V_t(J,M)
\end{pmatrix}\\
&\equiv -\log\boldsymbol{p}_{t,K}+\boldsymbol{u}_K+\beta\widetilde{\boldsymbol{S}}_K\boldsymbol{V}_{t+1}\boldsymbol{F}_w,
\end{aligned}$$

where $\log\boldsymbol{p}_{t,K}$ and $\boldsymbol{u}_K$ are analogously defined to $\boldsymbol{V}_t$. Taking the first difference of the equation above, we have the following recursive relationship, which is similar to (8)

$$\Delta\boldsymbol{V}_t = -\Delta\log\boldsymbol{p}_{t,K}+\beta\widetilde{\boldsymbol{S}}_K\boldsymbol{\Delta V}_{t+1}\boldsymbol{F}_w, \tag{A.12}$$

where $\Delta\boldsymbol{V}_t\equiv\boldsymbol{V}_t-\boldsymbol{V}_{t-1}$, and $\Delta\log\boldsymbol{p}_{t,K}$ is defined in the same way as $\boldsymbol{V}_t$. Multiplying $\boldsymbol{F}_w$ to both sides of the equation above,

$$\Delta\boldsymbol{V}_t\boldsymbol{F}_w = -\Delta\log\boldsymbol{p}_{t,K}F_w+\beta\widetilde{\boldsymbol{S}}_K\boldsymbol{\Delta V}_{t+1}\boldsymbol{F}_w\boldsymbol{F}_w.$$

Plugging matrices $\Delta\boldsymbol{V}_t\boldsymbol{F}_w$ and $\Delta\boldsymbol{V}_{t+1}\boldsymbol{F}_w$ from (A.11) into the equation above leads to the following moment condition with beliefs being the only unknowns:

$$\beta^{-1}(\boldsymbol{S}_i-\boldsymbol{S}_K)^{-1}\Delta\boldsymbol{\xi}_{t-1,i,K} = -\Delta\log\boldsymbol{p}_{t,K}\boldsymbol{F}_w+\widetilde{\boldsymbol{S}}_K(\boldsymbol{S}_i-\boldsymbol{S}_K)^{-1}\Delta\boldsymbol{\xi}_{t,i,K}\boldsymbol{F}_w.$$

Equivalently, we have

$$\left[\widetilde{\boldsymbol{S}}_K[\boldsymbol{S}_i-\boldsymbol{S}_K]^{-1}, \quad -\beta^{-1}[\boldsymbol{S}_i-\boldsymbol{S}_K]^{-1}\right]\left[\begin{array}{c}\Delta\boldsymbol{\xi}_{t,i,K}\boldsymbol{F}_w\\ \Delta\boldsymbol{\xi}_{t-1,i,K}\end{array}\right]=\Delta\log\boldsymbol{p}_{t,K}\boldsymbol{F}_w,$$

where $\left[\begin{array}{c}\Delta\boldsymbol{\xi}_{t,i,K}\boldsymbol{F}_w\\ \Delta\boldsymbol{\xi}_{t-1,i,K}\end{array}\right]$ is a $(2J-2)$ by M matrix.

A.5. Proof of Theorem 4

Proof of Lemma 1. Note that the *ex-ante* value function can be represented as

$$V(x,w)=-\log p_K(x,w)+\beta\sum_{x',w'}V(x',w')f(w'|w)s(x'|x,K).$$

With a slight abuse of notation, we have the following matrix representation,

$$V=-\log p_K+\beta\hat{S}_K V\boldsymbol{F}_w$$

where $V\equiv\{V(x=i,w=j)\}_{i,j}$ is a $J\times M$ matrix, p_K is defined analogously. $\hat{S}_K\equiv\{s(x'=j|x=i,K)\}_{i,j}$ is a $J\times J$ matrix. To derive a closed-form expression for the *ex-ante* value function, we first vectorize the matrix expression in the following.

$$\begin{aligned}vec(V)&=vec(-\log p_K+\beta\hat{S}_K V\boldsymbol{F}_w)\\&=vec(-\log p_K)+\beta[\boldsymbol{F}_w'\otimes\hat{S}_K]vec(V)\end{aligned}$$

As a result,

$$vec(V)=\left[I-\beta(\boldsymbol{F}_w'\otimes\hat{S}_K)\right]^{-1}vec(-\log p_K). \tag{A.13}$$

Note that $\left[I-\beta(\boldsymbol{F}_w'\otimes\hat{S}_K)\right]$ is invertible without imposing any restrictions.

A.6. Proof in Remark 3

We consider identification of subjective beliefs under an alternative assumption to Assumption 7 on the transition of state variables. In particular, the state evolution satisfies the following conditions.

Assumption A.6.1. (a) The observed state variables x and w evolve independently, i.e,

$$f(x',w'|x,w,a)=f(x'|x,a)f(w'|w,a)=f(x'|x)f(w'|w,a),$$

where $f(w'|w,a)$ is the w's evolution process .
(b) Agents believe that the state variables x and w evolve independently and have rational expectations on the evolution of w.

$$s(x',w'|x,w,a)=s(x'|x)s(w'|w,a)=s(x'|x)f(w'|w,a). \tag{A.14}$$

Under Assumption A.6.1, we can represent the log ratio of CCPs in period t as

$$\begin{aligned}\xi_{t,i,K}(x,w)\equiv&\left[u_i(x,w)+\beta\sum_{x',w'}V_{t+1}(x',w')f(w'|w,i)s(x'|x)\right]\\&-\left[u_K(x,w)+\beta\sum_{x',w'}V_{t+1}(x',w')f(w'|w,K)s(x'|x)\right].\\\equiv&\,u_i(x,w)-u_K(x,w)+\beta S(x)\boldsymbol{V}_{t+1}\left[F_i(w)-F_K(w)\right],\end{aligned} \tag{A.15}$$

where $S(x)\equiv[s(x'=1|x),\ldots,s(x'=J|x)]$, $\boldsymbol{V}_t$ is a J by $M-1$ matrix with its (k,j)-th element being $V_t(x=k,w=j)-V_t(x=k,w=M)$, and $F_a(w)\equiv[f(w'=1|w,a),\ldots,f(w'=M-1|w,a)]'$ excluding the state $w'=M$ because they sum up to be 1. The first difference of log ratio of CCPs is

$$\begin{aligned}\Delta\xi_{t,i,K}(x,w)&\equiv\xi_{t,i,K}(x,w)-\xi_{t-1,i,K}(x,w)\\&\equiv\beta S(x)\Delta\boldsymbol{V}_{t+1}\left[F_i(w)-F_K(w)\right],\end{aligned}$$

The matrix representation of the equation above is

$$\Delta\boldsymbol{\xi}_{t,i,K}\equiv\beta\boldsymbol{S}\Delta\boldsymbol{V}_{t+1}\left[\boldsymbol{F}_i-\boldsymbol{F}_K\right], \tag{A.16}$$

where $\Delta\boldsymbol{\xi}_{t,i,K}$ is a J by $M-1$ matrix with its (k,j)-th element being $\Delta\xi_{t,i,K}(x=k,w=j)$, $\boldsymbol{S}\equiv[S(x=1),S(x=2),\ldots,S(x=J)]'$, and $\boldsymbol{F}_i\equiv[F_i(w=1),\ldots,F_i(w=M-1)]$. We then represent the value function recursively by backward induction,

$$\boldsymbol{V}_t=-\log\boldsymbol{p}_{t,K}+\boldsymbol{u}_K+\beta\boldsymbol{S}\boldsymbol{V}_{t+1}\widetilde{\boldsymbol{F}}_K,$$

where $\widetilde{\boldsymbol{F}}_K$ is defined analogously to $\widetilde{\boldsymbol{S}}_K$, and the first difference of value function also has a recursive representation

$$\Delta \boldsymbol{V}_t = -\Delta \log \boldsymbol{p}_{t,K} + \beta \boldsymbol{S} \boldsymbol{\Delta V}_{t+1} \widetilde{\boldsymbol{F}}_K. \tag{A.17}$$

The derivation of the above equation is analogous to that in Eq. (A.12), so we skip the detail here.

To separate the unknown value function from the subjective beliefs, we need to impose the following rank conditions, which is similar to that in Assumption 4.

Assumption A.6.2. Both $\boldsymbol{S}$ and $\boldsymbol{F}_i - \boldsymbol{F}_K$ have full rank.

Note that $\boldsymbol{F}_i - \boldsymbol{F}_K$ is a known $M-1$ by $M-1$ matrix, so the full rank assumption is empirically testable. With this full rank condition, we can represent the first difference of the *ex ante* value function in the following closed-form representation of beliefs:

$$\Delta \boldsymbol{V}_{t+1} = \beta^{-1} \boldsymbol{S}^{-1} \Delta \boldsymbol{\xi}_{t,i,K} \left[\boldsymbol{F}_i - \boldsymbol{F}_K\right]^{-1}.$$

Plugging this closed-form representation of the first difference of *ex ante* value function into its recursive relationship and rearranging term leads to the following moment conditions with beliefs being the only unknowns:

$$\beta^{-1} \Delta \boldsymbol{\xi}_{t-1,i,K} \left(\boldsymbol{F}_i - \boldsymbol{F}_K\right)^{-1} = \boldsymbol{S} \left[-\Delta \log \boldsymbol{p}_{t,K} + \Delta \boldsymbol{\xi}_{t,i,K} \left(\boldsymbol{F}_i - \boldsymbol{F}_K\right)^{-1} \widetilde{\boldsymbol{F}}_K\right]. \tag{A.18}$$

Note that the equation above holds for $t = 3, 4, \ldots, T$. We stack all the $T-2$ equations and have

$$\boldsymbol{S} \underbrace{\begin{pmatrix} \left[-\Delta \log \boldsymbol{p}_{3,K} + \Delta \boldsymbol{\xi}_{3,i,K} \left[\boldsymbol{F}_i - \boldsymbol{F}_K\right]^{-1} \widetilde{\boldsymbol{F}}_K\right]' \\ \vdots \\ \left[-\Delta \log \boldsymbol{p}_{T,K} + \Delta \boldsymbol{\xi}_{T,i,K} \left[\boldsymbol{F}_i - \boldsymbol{F}_K\right]^{-1} \widetilde{\boldsymbol{F}}_K\right]' \\ [1, 1, \ldots, 1] \end{pmatrix}'}_{\boldsymbol{A}} = \underbrace{\begin{pmatrix} \beta^{-1} \left[\Delta \boldsymbol{\xi}_{3,i,K} \left(\boldsymbol{F}_i - \boldsymbol{F}_K\right)^{-1}\right]' \\ \vdots \\ \beta^{-1} \left[\Delta \boldsymbol{\xi}_{T,i,K} \left(\boldsymbol{F}_i - \boldsymbol{F}_K\right)^{-1}\right]' \\ [1, 1, \ldots, 1] \end{pmatrix}'}_{\boldsymbol{B}}, \tag{A.19}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are both J by $(T-2)(M-1)+1$ matrices, and $[1, 1, \ldots, 1]$ is a $1 \times J$ vector of ones, which is included to use the fact that every row of the beliefs $\boldsymbol{S}$ is a total probability so it adds up to be 1.

Assumption A.6.3. Matrix $\boldsymbol{A}$ has a full row rank.

Assumption A.6.3 implicitly requires that $(T-2)(M-1)+1 \geq J$. This assumption imposes restrictions to time period T and the possible values w takes, M. Under this assumption, the right inverse of matrix $\boldsymbol{A}$ exists and we denote it as $\boldsymbol{A}^+$. We apply the right inverse $\boldsymbol{A}^+$ to (A.19) to get a closed-form solution for $\boldsymbol{S}$:

$$\boldsymbol{S} = \boldsymbol{B}\boldsymbol{A}^+ \tag{A.20}$$

Theorem A.1. *Under Assumptions 1–3, A.6.1, A.6.2, and A.6.3, the subjective beliefs $s(x'|x)$ for $x, x' \in \{1, 2, \ldots, J\}$ are identified as a closed-form function of the CCPs $p_t(a|x, w)$ and the objective state transition $f_t(w'|w, a)$ for $t = 1, 2, \ldots, T$, where $T \geq \lceil * \rceil \frac{J-1}{M-1} + 2$.*

A.7. Proof of Theorem 5

The first-order Markov process $\{a_t, x_t, \tau\}$ indicates

$$\begin{aligned} \Pr(a_{t+}, x_{t+1}, a_t, x_t, a_{t-}) &= \sum_{\tau=1}^{H} \Pr(a_{t+}|x_{t+1}, \tau) \Pr(x_{t+1}, a_t|x_t, \tau) \Pr(\tau, x_t, a_{t-}), \\ &= \sum_{\tau=1}^{H} \Pr(a_{t+}|x_{t+1}, \tau) \Pr(a_t|x_{t+1}, x_t, \tau) \Pr(x_{t+1}|x_t, \tau) \Pr(\tau, x_t, a_{t-}), \end{aligned} \tag{A.21}$$

where $a_{t+} = h(a_{t+l}, \ldots, a_{t+1})$ and $a_{t-} = h(a_{t-1}, \ldots, a_{t-l})$.

Note that we have reduced the support of $a_{t+l}, \ldots, a_{t+1}$ to be the same as that of τ by the mapping $h(\cdot)$. We define the following matrices for given x_t, x_{t+1} and $a_t = k$,

$$\begin{aligned} M_{a_{t+}, x_{t+1}, k, x_t, a_{t-}} &= \left[\Pr(a_{t+} = i, x_{t+1}, k, x_t, a_{t-} = j)\right]_{i,j} \\ M_{a_{t+}, x_{t+1}, \tau} &= \left[\Pr(a_{t+} = i | x_{t+1}, \tau = j)\right]_{i,j} \\ M_{\tau, x_t, a_{t-}} &= \left[\Pr(\tau = i, x_t, a_{t-} = j)\right]_{i,j} \end{aligned}$$

$$D_{x_{t+1},k|x_t,\tau} = \mathrm{diag}\Big\{\Pr(x_{t+1},k|x_t,\tau=1),\ldots,\Pr(x_{t+1},k|x_t,\tau=L)\Big\}$$
$$D_{k|x_{t+1},x_t,\tau} = \mathrm{diag}\Big\{\Pr(k|x_{t+1},x_t,\tau=1),\ldots,\Pr(k|x_{t+1},x_t,\tau=L)\Big\}.$$

Eq. (A.21) can be rewritten as the following matrix form,

$$M_{a_{t+},x_{t+1},k,x_t,a_{t-}} = M_{a_{t+}|x_{t+1},\tau} D_{k|x_{t+1},x_t,\tau} D_{x_{t+1},k|x_t,\tau} M_{\tau,x_t,a_{t-}}. \tag{A.22}$$

Similarly, we have

$$M_{a_{t+},x_{t+1},x_t,a_{t-}} = M_{a_{t+}|x_{t+1},\tau} D_{x_{t+1}|x_t,\tau} M_{\tau,x_t,a_{t-}}, \tag{A.23}$$

where the matrices are defined analogously to those in (A.22) based on the following equality

$$\begin{aligned}
&\sum_{a_t=1}^{K} \Pr(a_{t+},x_{t+1},a_t,x_t,a_{t-}) \\
&= \sum_{\tau=1}^{H} \Pr(a_{t+}|x_{t+1},\tau)\Big[\sum_{a_t=1}^{K}\Pr(x_{t+1},a_t|x_t,\tau)\Big]\Pr(\tau,x_t,a_{t-}) \\
&= \sum_{\tau=1}^{H} \Pr(a_{t+}|x_{t+1},\tau)\Pr(x_{t+1}|x_t,\tau)\Pr(\tau,x_t,a_{t-}).
\end{aligned}$$

We use the first-order Markov property of $\{x_t, a_t\}$ to simply $\Pr(x_{t+1}|x_t,\tau,a_{t-})$ as $\Pr(x_{t+1}|x_t,\tau)$.

Under Assumption 10, the matrix $M_{a_{t+},x_{t+1},x_t,a_{t-}}$ for any given x_{t+1} and x_t has full rank. The equation above implies that $M_{a_{t+}|x_{t+1},\tau}$, $D_{x_{t+1}|x_t,\tau}$ and $M_{\tau,x_t,a_{t-}}$ are all invertible. We take inverse of (A.23) and multiply it from right to (A.22)

$$\begin{aligned}
M_{a_{t+},x_{t+1},k,x_t,a_{t-}} M^{-1}_{a_{t+},x_{t+1},x_t,a_{t-}} &= M_{a_{t+}|x_{t+1},\tau} D_{x_{t+1},k|x_t,\tau} D^{-1}_{x_{t+1}|x_t,\tau} M^{-1}_{a_{t+},x_{t+1},\tau} \\
&= M_{a_{t+}|x_{t+1},\tau} D_{k|x_{t+1},x_t,\tau} M^{-1}_{a_{t+},x_{t+1},\tau},
\end{aligned} \tag{A.24}$$

The equation above shows an eigenvalue–eigenvector decomposition of an observed matrix on the left-hand side. Assumptions 11 and 12 guarantee that this decomposition is unique. Therefore, the eigenvector matrix $M_{a_{t+}|x_{t+1},\tau}$, i.e., $\Pr(a_{t+}|x_{t+1},\tau)$ is identified. We can recover the matrix $M_{\tau,x_t,a_{t-}}$ from (A.23). The distribution $f(x_{t+1},a_t|x_t,\tau)$, and therefore $\Pr(a_t|x_t,\tau) = p_t(a_t|x_t,\tau)$ by integrating out x_{t+1}, can then be identified from Eq. (A.21) due to the invertibility of matrix $M_{a_{t+},x_{t+1},\tau}$.

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