Supplementary appendix (not intended for publication): sufficient conditions for nonexistence of surplus coalitions in partitional equilibria

In this appendix, we provide sufficient condition for the nonexistence of surplus coalitions when the ideological positions of the legislators are known and the uncertainty is about their ideological intensities. Suppose θ_i is distributed with cumulative distribution function F_i on $[\underline{\theta}_i, \overline{\theta}_i]$. Assume that F_i has a differentiable density function denoted by f_i. Further assume that each legislator i's $(i = 1, 2)$ message rule takes a partition form as in Crawford and Sobel (1982), i.e., legislator i sends message m_i^k if $\theta_i \in (\theta_i^k, \theta_i^{k+1}]$ $(k = 1, ..., K_i)$ where $\underline{\theta}_i = \theta_i^1 < \theta_i^2 < ... < \theta_i^{K_i+1} = \overline{\theta}_i$.

Proposition 1. *Suppose* $\frac{f_i(\theta)}{F_i(b)-F_i(\theta)}$ *is weakly increasing in* θ *for all* $b > \theta$. *Then any equilibrium proposal with* $x \neq \tilde{x}$ *that is accepted with positive probability in equilibrium must have* $x_i > 0$ *for no more than one legislator* $i \neq 0$.

Proof: Let $z = (y; x)$ be an equilibrium proposal for some $m = (m_1^k, m_2^\ell)$. We show that it is not possible to have $x_1 > 0$ and $x_2 > 0$ in equilibrium. Suppose to the contrary that an interior maximum exists for the chair's problem

$$z \in \arg\max_{z'=(y';x')} (c - x_1' - x_2' + \theta_0 v(y', \hat{y}_0))\beta(z'|m) + \theta_0 v(\tilde{y}, \hat{y}_0)\left(1 - \beta(z'|m)\right). \quad (1)$$

where

$$\beta(z|m) = 1 - (1 - \beta_1(z|m_1))\left(1 - \beta_2(z|m_2)\right) \quad (2)$$

is the conditional probability that x is accepted and $\beta_i(z|m_i)$ is the conditional acceptance probability for legislator i. Legislator i votes for z if $x_i + \theta_i v(y, \hat{y}_i) \geq \theta_i v(\tilde{y}, \hat{y}_i)$, that is, if $\theta_i \leq \frac{x_i}{v(\tilde{y},\hat{y}_i)-v(y,\hat{y}_i)}$. Letting $\theta_i(z) = \frac{x_i}{v(\tilde{y},\hat{y}_i)-v(y,\hat{y}_i)}$, we have

$$\beta_i(z|m_i^k) = \frac{F_i(\theta_i(z)) - F_i(\theta_i^k)}{F_i(\theta_i^{k+1}) - F_i(\theta_i^k)}. \quad (3)$$

First order necessary conditions for an interior maximum include:

$$-\beta(z|m) + [c - x_1 - x_2 + \theta_0(v(y, \hat{y}_0) - v(\tilde{y}, \hat{y}_0))]\frac{\partial \beta(z|m)}{\partial x_i} = 0$$

for $i = 1, 2$. If $c - x_1 - x_2 + \theta_0(v(y, \hat{y}_0) - v(\tilde{y}, \hat{y}_0)) = 0$, then $\beta(z|m) = 0$ and therefore the chair's payoff is not maximized. So $c - x_1 - x_2 + \theta_0(v(y, \hat{y}_0) - v(\tilde{y}, \hat{y}_0)) > 0$ and rearranging terms, we have

$$\frac{\partial\beta(z|m)}{\partial x_1} = \frac{\partial\beta(z|m)}{\partial x_2}. \tag{4}$$

Substituting (4) in the second order necessary conditions, we must have

$$[-2\frac{\partial\beta(z|m)}{\partial x_1} + (c - x_1 - x_2 - x_0^*)\frac{\partial^2\beta(z|m)}{\partial x_1^2}][-2\frac{\partial\beta(z|m)}{\partial x_1} + (c - x_1 - x_2 - x_0^*)\frac{\partial^2\beta(z|m)}{\partial x_2^2}]$$
$$-[-2\frac{\partial\beta(z|m)}{\partial x_1} + (c - x_1 - x_2 - x_0^*)\frac{\partial^2\beta(z|m)}{\partial x_1 \partial x_2}]^2 \geq 0$$

where $x_0^* = \theta_0(v(y, \hat{y}_0) - v(\tilde{y}, \hat{y}_0))$.
This is not possible if $\frac{\partial^2\beta(z|m)}{\partial x_1 \partial x_2} > \frac{\partial^2\beta(z|m)}{\partial x_i^2}$ for $i = 1, 2$. Substituting for $\beta(z|m)$ from (2) and (3), the latter inequalities are reduced to

$$-g_1(\theta_1(z))g_2(\theta_2(z)) < g_i'(\theta_i(z))(1 - G_j(\theta_j(z))), \tag{5}$$

for $i = 1, 2$, $j \neq i$, where G_1 is the distribution of θ_1 conditional on $\theta_1 \in (\theta_1^k, \theta_1^{k+1}]$, G_2 is the distribution of θ_2 conditional on $\theta_2 \in (\theta_2^\ell, \theta_2^{\ell+1}]$, and g_1 and g_2 are the associated densities. From (4), we have $g_1(\theta_1(z))(1 - G_2(\theta_2(z))) = g_2(\theta_2(z))(1 - G_1(\theta_1(z)))$. Substituting in (5), we must have $g_i'(\theta_i(z))(1 - G_i(\theta_i(z)) + g_i(\theta_i(z))^2 > 0$ for $i = 1, 2$. Since $f_i(\theta_i)/(F_i(b) - F_i(\theta_i))$ is weakly increasing in θ_i for any θ_i and b in the support of F_i with $\theta_i < b$, we have $\ln(f_i(\theta_i)/(F_i(b) - F_i(\theta_i))$ is also weakly increasing, and so $f_i'(\theta_i)/f_i(\theta_i) + f_i(\theta_i)/(F_i(b) - F_i(\theta_i)) \geq 0$. Substituting in conditional distributions, it is not possible to satisfy second order necessary conditions for an interior maximum.
∎