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Uniqueness of Stationary Equilibrium Payoffs in the Baron–Ferejohn Model

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We consider a multilateral sequential bargaining model in which the players may differ in their probability of being selected as the proposer and the rate at which they discount future payoffs. For games in which agreement requires less than unanimous consent, we characterize the set of stationary subgame perfect equilibrium payoffs. With this characterization, we establish the uniqueness of the equilibrium payoffs. For the case where the players have the same discount factor, we show that the payoff to a player is nondecreasing in his probability of being selected as the proposer. For the case where the players have the same probability of being selected as the proposer, we show that the payoff to a player is nondecreasing in his discount factor. *Journal of Economic Literature* Classification numbers: C72, C78, D70. © 2001 Elsevier Science (USA)

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1. INTRODUCTION

In their seminal contribution, Baron and Ferejohn [2] present a simple sequential model of multilateral bargaining with majority rule. The game they consider is a standard "divide the dollar" game where n risk neutral players are randomly selected or "recognized" to make proposals as to how to divide a fixed cake and agreement requires the consent of a simple majority. Baron and Ferejohn [2] show that any division of the cake can be supported as a subgame perfect equilibrium if there are at least 5 players and the (common) rate at which players discount future payoffs is sufficiently high. In light of this result, they restrict attention to stationary strategies. While in their model they allow for the probabilities with which players are selected to be the proposer to differ, they only establish the

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uniqueness of the stationary subgame perfect equilibrium when these recognition probabilities are restricted to be the same.[2]

Baron and Ferejohn [2] also show with an example that when the players have different probabilities of being selected as proposer, the equilibrium need not be unique. In particular, they construct an example with a continuum of equilibria. However, in this example all the equilibria yield the same payoffs.

In this paper, we extend the Baron and Ferejohn [2] model to general q-quota agreement rules and allow the discount factors to differ across players. We show that, for general recognition probabilities, the vector of stationary subgame perfect equilibrium payoffs is unique. The model we consider is a special case of Banks and Duggan [1], who establish the existence of stationary subgame perfect equilibria when the set of alternatives is multidimensional and players are risk averse. Note, however, that the uniqueness result obtained here does not necessarily extend to the more general environment as shown by Banks and Duggan [1].

The paper is organized as follows. In the next section, we introduce the model and define the basic concepts. In Section 3, we characterize the set of stationary subgame perfect equilibria. In Section 4, we establish the existence of stationary subgame perfect equilibria. In Section 5, we establish certain monotonicity properties of the equilibrium payoffs. We show that, when the players have a common discount factor, the equilibrium payoffs are monotone nondecreasing in the recognition probabilities. Furthermore, for the case where the players have equal recognition probabilities, we show that the equilibrium payoffs are monotone nondecreasing in the discount factors. In Section 6, we prove the uniqueness of stationary subgame perfect equilibrium payoffs.

2. MODEL

We consider a sequential bargaining game with complete information, linear preferences and q-quota majority rule. The model is as follows. Let $N=\{1, ..., n\}$ denote the set of players with typical elements i and j, where $n \geqslant 2$. The players are to distribute a unit of perfectly divisible cake among themselves. Let X denote the set of feasible allocations, that is, $X=\{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i \leqslant 1\}$, where x_i denotes the cake share for player i. We assume each player has a linear utility function that depends on his cake share only. We also assume that players discount the future and we let $\delta_i < 1$ denote player i's discount factor.

[2] An interesting feature of this equilibrium is that, as shown by Baron and Kalai [3], it is the unique simplest equilibrium of the corresponding automaton game.

The game is played as follows. At date 0, player i is selected as the proposer with probability $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$. We will refer to p_i as the recognition probability of player i. Upon his recognition, player i offers an allocation in X. Each player sequentially responds by either accepting or rejecting the proposal. If at least $q \in \{1, ..., n\}$ people including the proposer accept the proposal, the game ends and the cake is shared according to the accepted proposal. If not, the process moves to date 1 and the procedure is repeated except that a new proposer may be selected. This process continues until an allocation is accepted. If $x \in X$ is accepted at date $t \in \{0, 1, ...\}$, player i's payoff is given by $\delta_i^t x_i$. If no allocation is ever accepted, each player receives a payoff of zero.

The rest of the section describes the solution concept that we use. Let h^t denote the past history (identity of previous proposers, what proposals they made, how each player voted for these proposals) together with the identity of the current proposer and the proposal he made if he made one. A feasible action for player i at date t is denoted by $a_i^t(h^t)$. Given any set S, let $\Delta(S)$ denote the set of probability measures on S. When i is the proposer $a_i^t(h^t) \in \Delta(X)$ denotes the (mixed) proposal offered by i at date t, when the history is h^t. When i is not the proposer $a_i^t(h^t) \in \Delta(\{\text{accept}, \text{reject}\})$ denotes the (mixed) decision rule by player i as to whether to accept or reject the current proposal at date t, when the history is h^t. A strategy s_i for player i is a sequence of actions $\{a_i^t(h^t)\}_{t=1}^{\infty}$, and a strategy profile s is an n-tuple of strategies, one for each player.

A strategy profile is subgame perfect if and only if no player can benefit by deviating from his strategy at a single date (see for example Fudenberg and Tirole [4]). A strategy profile is stationary if it does not depend on the current date and past history. A strategy profile is stationary subgame perfect (SSP) if it is stationary and subgame perfect. An SSP outcome and payoff are the outcome and the payoff generated by an SSP strategy profile.

It is well known that in multilateral bargaining games like the one considered here there is multiplicity of subgame perfect equilibria even under unanimity rule (see, for example, Sutton [6]). However, it has also been recognized that stationarity is typically able to select a unique equilibrium (see, for example, Baron and Ferejohn [2] and Merlo and Wilson [5]). Thus, we restrict our attention to SSP equilibria.

3. CHARACTERIZATION OF SSP EQUILIBRIA

In this section we characterize the SSP payoffs and strategies.[3] To simplify notation we define M^i as the set of n-dimensional real vectors such

[3] As it is customary in the literature, we assume a player votes in favor of a proposal whenever he is indifferent between accepting and rejecting it.

that the ith component is zero. We also let $e \in \mathbb{R}^n$ denote the n-dimensional unit vector. Finally we let $D = [d_{ij}]$ denote the $n \times n$ diagonal matrix with $d_{ii} = \delta_i$. The following theorem characterizes the set of SSP equilibria.

THEOREM 1. *The set of SSP equilibria can be described as follows. Let $i \in N$ denote the proposer. Then*

$$a_i = (x_i^i, x_j^i) \qquad \text{with probability } f_i(\gamma_i)$$

$$a_j = \text{accept iff } x_j^i \geqslant \delta_i v_j \qquad \text{for all} \quad j \neq i,$$

where

$$x_i^i = 1 - \sum_{j \neq i} \delta_j v_j \gamma_{ij},$$

$$x_j^i = \delta_j v_j \gamma_{ij},$$

and for all $k \in N$

$$v_i = p_i \left(1 - \sum_{j \neq i}^{n} \delta_j v_j \sum_{\gamma_i \in C_i} \gamma_{ij} f_i(\gamma_i) \right) + \delta_i v_i \sum_{j \neq i}^{n} p_j \sum_{\gamma_j \in C_j} \gamma_{ji} f_j(\gamma_j), \tag{1}$$

$$\gamma_i = (\gamma_{i1}, ..., \gamma_{in})' \in C_i,$$

and C_k is the set of minimizers for the following program

$$\min_z \quad z' D v \tag{2}$$

$$\text{subject to} \quad z' e = q - 1$$

$$z \in \{0, 1\}^n \cap M^k$$

and $v = (v_1, ..., v_n)'$.

Proof. Let v be a stationary payoff vector. Then the continuation payoff for player i is given by $\delta_i v_i$. Given a proposal x^j by player j, player $i \neq j$ will accept the proposal if $x_i^j \geqslant \delta_i v_i$ and will reject it if $x_i^j < \delta_i v_i$. Suppose now player i is the proposer. Switching the roles of i and j in the argument above, player i needs to give player j at least $\delta_j v_j$ in order to induce acceptance by player j. Since the proposer's payoff is strictly increasing in his own cake share, he will give player j exactly $\delta_j v_j$ to buy player j's vote.

Note that the proposer needs $q - 1$ votes in addition to his vote in order to induce acceptance of his proposal. Thus, when player i is the proposer, choosing a payoff maximizing proposal that will be accepted is equivalent to solving

$$\max_{(z_j)_{j\neq i}} \quad 1-\sum_{j\neq i} z_j\delta_j v_j$$

$$\text{subject to} \quad \sum_{j\neq i} z_j = q-1$$

$$z_j \in \{0, 1\}$$

or equivalently solving the problem (2).

Let C_i denote the set of minimizers for (2). Clearly an SSP mixed strategy for players puts positive mass only on the set C_i. Note that player i's payoff is the same for all $\gamma_i \in C_i$ and hence he is indifferent as to which $q-1$ players he pays off. In equilibrium, the probability distributions $(f_1, ..., f_n)$ must induce the stationary payoff vector v. Next we show that this is satisfied by (1).

Consider the situation at the beginning of the current period, before the identity of the proposer has been revealed. With probability p_i player i is the proposer, in which case the payoff to him is $1-\sum_{j\neq i}^{n} \delta_j v_j \sum_{\gamma_i \in C_i} \gamma_{ij} f_i(\gamma_i)$ given his SSP strategy. With probability p_j, player $j\neq i$ is the proposer in which case the expected payoff to player i is $\delta_i v_i \sum_{\gamma_j \in C_j} \gamma_{ji} f_j(\gamma_j)$. Since this is true for all $j\neq i$, the right hand side of (1) gives the expected payoff to player i at the beginning of current period before the identity of the proposer is revealed.

The proof follows from the preceding discussion noting that a deviation by players from the strategy described at a single date does not affect the continuation payoffs. Q.E.D.

Note that given the SSP payoff vector v, an SSP proposal x^i by player i can be identified by the $(n-1)$-dimensional vector $\gamma_i \in C_i \subset \{0, 1\}^n \cap M^i$ which specifies the players whose votes are bought by player i. Intuitively, under the proposal corresponding to γ_i, player $j\neq i$ receives his continuation payoff if $\gamma_{ij}=1$ and he receives nothing if $\gamma_{ij}=0$. We define a coalition partner of player i as a player whose vote is bought by player i, that is. a player who receives his continuation payoff when i is the proposer. With the identification above, a stationary (mixed) strategy for player i is a probability distribution $f_i \in \Delta(C_i)$. An SSP strategy f_i for player i induces offer probabilities

$$r_{ij} = \sum_{\gamma_i \in C_i} \gamma_{ij} f_i(\gamma_i),$$

where r_{ij} denotes the probability that j is a coalition partner of player i. Thus, we can rewrite (1) in terms of the offer probabilities r_{ij} as

$$v_i = p_i \left(1 - \sum_{j \neq i}^{n} r_{ij} \delta_j v_j \right) + \delta_i v_i \sum_{j \neq i}^{n} p_j r_{ji}. \tag{3}$$

Let $r_i = (r_{i1}, ..., r_{in})$ denote the vector of offer probabilities for player i. Note that $\gamma_{ij} = 0$ for all $\gamma_i \in C_i$ and hence $r_{ii} = 0$ which in turn implies that $r_i \in M^i$. Let $r = (r_1, ..., r_n)$ denote the collection of offer probability vectors. Given r, we can write the payoff vector $v = (v_1, ..., v_n)$ as a fixed point of the operator $A(\cdot\,; r)$: $[0, 1]^n \to [0, 1]^n$ where

$$A_i(v; r) = p_i \left(1 - \sum_{j \neq i}^{n} r_{ij} \delta_j v_j \right) + \delta_i v_i \sum_{j \neq i}^{n} p_j r_{ji} \tag{4}$$

for all i.

The next theorem characterizes the SSP payoffs. The proof of this theorem relies on a lemma (Lemma 1) which is stated and proved in the Appendix.

THEOREM 2. *A payoff vector v is SSP if and only if there exists offer probabilities r such that*

(i) *given v, for all $i \in N$, r_i, is a minimizer of*

$$\begin{aligned} \min_{z} \quad & z' D v \\ \textit{subject to} \quad & z' e = q - 1 \\ & z \in [0, 1]^n \cap M^k \end{aligned} \tag{5}$$

(ii) *given r, v is a fixed point of $A(\cdot\,; r)$.*

Proof. By Theorem 1 and the arguments following that, it suffices to show that for all i, r_i is a minimizer of (5) if and only if it is induced by an SSP strategy f_i for player i.

Recall that C_i is the set of minimizers for (2). Also, note that the constraint set for (5) is the convex hull of constraint set of (2). Therefore by Lemma 1 in the Appendix, r_i solves (5) if and only is it is in the convex hull of C_i. Thus, r_i solves (5) if and only if there exists a probability distribution f_i with support C_i such that

$$r_{ij} = \sum_{\gamma_i \in C_i} \gamma_{ij} f_i(\gamma_i),$$

that is, r_i is induced by f_i. The probability distributions $(f_1, ..., f_n)$ that induce r must also induce v by the hypothesis that v is a fixed point of $A(\cdot; r)$. Thus, the proof follows. Q.E.D.

Note that, any SSP equilibrium proposal can be uniquely identified with a pair (v, r) where v is the equilibrium payoff vector and r is the corresponding offer probability collection. In what follows, we will refer to such a pair as an equilibrium outcome. Also note that, to simplify notation we have suppressed the interdependency between the payoff vector v and offer probability collection r. Since this interdependency plays a crucial role in the argument we use in the next section to establish the existence of equilibria, we will drop this simplification there.

4. EXISTENCE OF SSP EQUILIBRIA

In this section we prove the existence of SSP equilibria. We show the existence of SSP payoffs in the $(n-1)$-dimensional unit simplex and in the rest of the paper we show the uniqueness of the SSP payoffs in this set.[4]

For all $i \in N$, let $\Sigma_i = [0, 1]^n \cap M^i$ and let $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$.

THEOREM 3. *There exists an SSP equilibrium outcome.*

Proof. Let $\bar{r}_i(v)$ denote the set of minimizers for problem (5) given v and let $\bar{r}_1(v) \times \cdots \times \bar{r}_n(v)$. By the Theorem of the Maximum, $\bar{r}(v)$ is upper hemi-continuous, compact valued and nonempty for all v. It can be easily verified that $\bar{r}(v)$ is convex valued for all v. Also note that, for fixed r, $A(\cdot; r)$ is a contraction mapping and that maps the $(n-1)$-dimensional unit simplex to itself. Thus it has a unique fixed point $v(r)$ which is a continuous function of r. Hence the correspondence $\bar{r} \circ v = \Gamma\colon \Sigma \to \Sigma$ defined by

$$\Gamma(r) = \{r' \in \Sigma : r' \in \bar{r}(v(r))\}$$

is also compact and convex valued, upper hemi-continuous and nonempty for all $r \in \Sigma$. By Theorem 2, $(v(\bar{r}), \bar{r})$ is an SSP equilibrium outcome if and only if $\bar{r}$ is a fixed point of Γ. Hence by Kakutani's Fixed Point Theorem the result follows. Q.E.D.

[4] An earlier version of the paper shows that any SSP payoff must lie in the $(n-1)$-dimensional unit simplex. That the payoffs must sum to unity follows by the definition of the operator $A(\cdot; r)$. Since it is intuitively obvious that an SSP payoff must be nonnegative, we avoid the cumbersome proof and restrict our attention directly to the nonnegative payoffs.

5. MONOTONICITY OF SSP PAYOFFS

In this section, we show that any SSP payoff vector must satisfy certain monotonicity conditions. These conditions are used in the next section to establish the uniqueness of SSP payoffs. Furthermore, we show that, when the players have a common discount factor, if player *i*'s recognition probability is not larger than player *j*'s, then his payoff cannot be larger than player *j*'s payoff. Similarly, when the players have equal recognition probabilities, if player *i*'s discount factor is not larger than player *j*'s, then his payoff cannot be larger than player *j*'s payoff.

Given an SSP payoff vector v, let w_i denote the optimized value of the objective function of (5), which is also equal to the optimized value of the objective function of (2). Intuitively w_i is the sum of the cake shares disbursed by player i to his coalition partners in order to reach agreement on his proposal. i.e. w_i is the cost of the cheapest coalition when i is the proposer. We refer to w_i as the disbursement by player i and to the vector $w=(w_1, ..., w_n)$ as a disbursement vector.

To simplify the notation, let μ_i be defined as

$$\mu_i = \sum_{k \neq i} p_k r_{ki},$$

that is, μ_i is the probability that player i is in the winning coalition when he is not the proposer. Note that, by definition, $0 \leqslant \mu_i \leqslant 1 - p_i$, and hence,

$$0 < \frac{p_i}{1-\delta_i \mu_i} < 1, \tag{6}$$

for all $i \in N$. Let $\mu = (\mu_1, ..., \mu_n)$. As before, we economize on notation and suppress the dependency of w and μ on the payoff vector v.

Rearranging (3) and using the definitions of w_i and μ_i we can write v_i as

$$v_i = \frac{p_i(1-w_i)}{1-\delta_i \mu_i} \tag{7}$$

or equivalently,

$$v_i = p_i(1-w_i) + \delta_i \mu_i v_i. \tag{8}$$

Throughout this section we enumerate N as $\{i_1, ..., i_n\}$ such that

$$\delta_{i_1} v_{i_1} \leqslant \cdots \leqslant \delta_{i_n} v_{i_n}.$$

This enumeration implies that there are (at least) $q-1$ players whose, votes are not more expensive than player i_q's vote. Hence, in deciding whether to buy player j's vote or not, the proposer only needs to compare the cost of player j's vote to the cost of player i_q's vote.

In equilibrium, the following conditions must hold. First,

$$\mu_j = \begin{cases} 1 - p_j & \text{if} \quad \delta_j v_j < \delta_{i_q} v_{i_q}, \\ 0 & \text{if} \quad \delta_j v_j > \delta_{i_q} v_{i_q} \end{cases} \tag{9}$$

and $\mu_j \leqslant 1 - p_j$ if $\delta_i v_i = \delta_{i_q} v_{i_q}$.

This follows immediately from the fact that, for each player $k \in N$, r_k solves (5). The intuition behind the condition is as follows. Let j be a player such that $\delta_j v_j < \delta_{i_q} v_{i_q}$. Let $k \neq j$ be the proposer and consider the situation faced by the proposer. To induce the acceptance of his proposal he needs to buy the votes of the least costly $q-1$ other players by offering them their continuation payoffs. Since $\delta_j v_j < \delta_{i_q} v_{i_q}$, player j is in this group with certainty. Thus, the offer probability of player j is always 1 for all $k \neq j$. Since player j is always in the winning coalition when he does not propose, and since the probability that he does not propose is $1 - p_j$, the probability that he is in the winning coalition when someone else proposes is $1 - p_j$. On the other hand, if $\delta_j v_j > \delta_{i_q} v_{i_q}$, any proposer $k \neq j$ can do better by not buying the vote of player j as there are always $q-1$ other votes that are cheaper to buy. That is, the probability that player j is in the winning coalition when someone else proposes is 0. Note that, by (9) $\mu_j > 0$ implies $\delta_j v_j \leqslant \delta_{i_q} v_{i_q}$, and $\mu_j < 1 - p_j$ implies $\delta_j v_j \geqslant \delta_{i_q} v_{i_q}$.

Second,

$$w_j = \begin{cases} w_{i_q} = \sum_{k=1}^{q-1} \delta_{i_k} v_{i_k} & \text{if} \quad \delta_j v_j \geqslant \delta_{i_q} v_{i_q}, \\ w_{i_q} + \delta_{i_q} v_{i_q} - \delta_j v_j & \text{if} \quad \delta_j v_j \leqslant \delta_{i_q} v_{i_q}. \end{cases} \tag{10}$$

To understand this condition, note that, given the ordering $\delta_{i_q} v_{i_1} \leqslant \cdots \leqslant \delta_{i_n} v_{i_n}$ of the payoffs, the votes of players $i_1, ..., i_{q-1}$ are always the cheapest to buy. Hence, the disbursement by player j, for $j = i_q, ..., i_n$, is $\sum_{k=1}^{q-1} v_{i_k}$. But a player cannot buy his own vote as he needs $q-1$ votes in addition to his vote. Then, the next best choice for player j, when $\delta_j v_j \leqslant \delta_{i_q} v_{i_q}$, is to buy the votes of players $\{i_1, ..., i_q\} - \{j\}$. In other words, the disbursement, by player j and the disbursement by player q differ by $\delta_{i_q} v_{i_q} - \delta_j v_j$. This explains condition (10).

Third, using (9) and (10), we can write (7) as

$$v_j = \begin{cases} p_j (1 - w_{i_q}) & \text{if} \quad \delta_j v_j > \delta_{i_q} v_{i_q}, \\ \dfrac{p_j}{1 - \delta_j} (1 - w_{i_q} - \delta_{i_q} v_{i_q}) & \text{if} \quad \delta_j v_j < \delta_{i_q} v_{i_q}. \end{cases} \tag{11}$$

Also, if $\delta_j v_j \geqslant \delta_{i_q} v_{i_q}$

$$p_j \delta_j (1 - w_{i_q}) \leqslant \delta_j v_j \leqslant \frac{p_j \delta_j}{1 - \delta_j + p_j \delta_j} (1 - w_{i_q}) \tag{12}$$

since $1 - p_j \geqslant \sum_{k \neq j} r_{kj} p_k \geqslant 0$.

Finally, if $\delta_j v_j < \delta_{i_q} v_{i_q} \leqslant \delta_k v_k$, then

$$\delta_k v_k > \frac{p_j \delta_j}{1 - \delta_j + p_j \delta_j} (1 - w_{i_q}). \tag{13}$$

To see this condition holds in equilibrium, note that,

$$\begin{aligned} \delta_k v_k &> \delta_j v_j \\ &= \frac{p_j \delta_j}{1 - \delta_j} (1 - w_{i_q}) - \frac{p_j \delta_j}{1 - \delta_j} (\delta_{i_q} v_{i_q}) \\ &\geqslant \frac{p_j \delta_j}{1 - \delta_j} (1 - w_{i_q}) - \frac{p_j \delta_j}{1 - \delta_j} (\delta_k v_k). \end{aligned}$$

Rearranging, we obtain (13).

THEOREM 4. *Suppose v is an SSP payoff vector. Let $\{i_1, ..., i_n\}$ be an enumeration of N such that*

$$\delta_{i_1} v_{i_1} \leqslant \cdots \leqslant \delta_{i_n} v_{i_n}.$$

Then,

(i) *If $\delta_j v_j \leqslant \delta_{i_q} v_{i_q} < \delta_k v_k$, then $p_j \delta_j < p_k \delta_k$.*

(ii) *If $\delta_j v_j < \delta_k v_k \leqslant \delta_{i_q} v_{i_q}$, then $\frac{p_j \delta_j}{1 - \delta_j} < \frac{p_k \delta_k}{1 - \delta_k}$.*

(iii) *If $\delta_k v_k \geqslant \delta_{i_q} v_{i_q}$ and $p_j \delta_j \leqslant p_k \delta_k$ then $\delta_k v_k \geqslant \delta_j v_j$.*

(iv) *If $\delta_k v_k < \delta_{i_q} v_{i_q}$ and $\frac{p_j \delta_j}{1 - \delta_j} \leqslant \frac{p_k \delta_k}{1 - \delta_k}$ then $\delta_k v_k \geqslant \delta_j v_j$.*

Proof. (i) If $\delta_j v_j < \delta_{i_q} v_{i_q}$, then, substituting for v_k from (11) in (13), we obtain

$$p_k \delta_k (1 - w_{i_q}) > \frac{p_j \delta_j}{1 - \delta_j + p_j \delta_j} (1 - w_{i_q}).$$

The result follows since $1 - \delta_j + p_j \delta_j < 1$. If, on the other hand, $\delta_j v_j = \delta_{i_q} v_{i_q}$ then, by (11) and (12), $\delta_k v_k = p_k \delta_k (1 - w_{i_q}) > \delta_j v_j \geqslant p_j \delta_j (1 - w_{i_q})$ from which the desired result follows immediately.

(ii) If $\delta_k v_k < \delta_{i_q} v_{i_q}$, then the result follows immediately, since by (11),

$$\frac{p_j\delta_j}{1-\delta_j}(1-w_{i_q}-\delta_{i_q}v_{i_q})=\delta_j v_j<\delta_k v_k=\frac{p_k\delta_k}{1-\delta_k}(1-w_{i_q}-\delta_{i_q}v_{i_q}).$$

If, on the other hand, $\delta_k v_k = \delta_{i_q} v_{i_q}$, then, from (12) and (13),

$$\frac{p_j\delta_j}{1-\delta_j+p_j\delta_j}(1-w_{i_q})<\delta_k v_k\leqslant\frac{p_k\delta_k}{1-\delta_k+p_k\delta_k}(1-w_{i_q}),$$

which implies the desired result.

(iii) Let $p_j\delta_j \leqslant p_k\delta_k$ and suppose on the contrary $\delta_k v_k < \delta_j v_j$. Since $\delta_k v_k \geqslant \delta_{i_q} v_{i_q}$, we have $\delta_j v_j > \delta_{i_q} v_{i_q}$. Then, by (11) and (12)

$$\delta_j v_j = p_j\delta_j(1-w_{i_q}) > \delta_k v_k \geqslant p_k\delta_k(1-w_{i_q}).$$

But then $p_j\delta_j > p_k\delta_k$. This contradiction proves the result.

(iv) Let $\frac{p_j\delta_j}{1-\delta_j} \leqslant \frac{p_k\delta_k}{1-\delta_k}$ and suppose on the contrary $\delta_k v_k < \delta_j v_j$. If $\delta_{i_q} v_{i_q} \geqslant \delta_j v_j$, then by part (ii) of the Lemma, $\frac{p_j\delta_j}{1-\delta_j} > \frac{p_k\delta_k}{1-\delta_k}$ which is a contradiction. If, instead, $\delta_j v_j > \delta_{i_q} v_{i_q}$, then, by (12) and (13),

$$\frac{p_j\delta_j}{1-\delta_j+p_j\delta_j}(1-w_{i_q})\geqslant\delta_j v_j=\frac{p_k\delta_k}{1-\delta_k+p_k\delta_k}(1-w_{i_q}).$$

Rearranging the above expression, we obtain $\frac{1-p_j}{1-\delta_j} > \frac{p_k\delta_k}{1-\delta_k}$ contradiction. The proof follows by contradiction of both possible cases. Q.E.D.

Note that, when the players have a common discount factor, $p_j\delta_j \leqslant p_k\delta_k$ if and only if $\frac{p_j\delta_j}{1-\delta_j} \leqslant \frac{p_k\delta_k}{1-\delta_k}$ and both of these conditions are equivalent to $p_j \leqslant p_k$. Thus, parts (iii) and (iv) of Theorem 4 imply that when the players have a common discount factor, the SSP payoffs are monotone non-decreasing in the recognition probabilities.

Corrolary 1. *Let v be an SSP payoff function and suppose $\delta_i = \delta$ for all $i \in N$. Then, $p_j \leqslant p_k$ implies $v_j \leqslant v_k$.*

Similarly, when the players have equal recognition probabilities, $p_j\delta_j \leqslant p_k\delta_k$ if and only if $\frac{p_j\delta_j}{1-\delta_j} \leqslant \frac{p_k\delta_k}{1-\delta_k}$ and both of these conditions are equivalent to $\delta_j \leqslant \delta_k$. Thus, parts (iii) and (iv) of Theorem 4 imply that when the players have equal recognition probabilities, the SSP payoffs are monotone nondecreasing in the discount factors.[5]

Corollary 2. *Let v be an SSP payoff function and suppose $p_i = 1/n$ for all $i \in N$. Then, $\delta_j \leqslant \delta_k$ implies $v_j \leqslant v_k$.*

The following example illustrates that when the discount factor is the same, the players can have equal SSP payoffs even if their recognition probabilities are different. There are 3 players, a common discount factor $\delta = 0.95$, recognition probabilities $p_1 = 0.45$, $p_2 = 0.4$, and $p_3 = 0.15$, and the agreement rule is majority rule. In this example $v_i = 1/3$, $i = 1, 2, 3$. Deriving the necessary conditions for the strict monotonicity of the SSP payoffs in the recognition probabilities is complicated even when there are only three players. The following is an example of sufficient conditions for strict monotonicity. All players have a common discount factor δ, and the recognition probabilities are p_1, for player 1, and $(1 - p_1)/(n - 1)$ for all other players. Then, $v_1 > v_2$ if and only if $p_1 > \frac{1}{n-(q-1)\delta}$ and $v_1 < v_2$ if and only if $p_1 < \frac{1-\delta}{n-q\delta}$.

6. UNIQUENESS OF SSP PAYOFFS

In this section, we prove the uniqueness of SSP payoff vectors. Baron and Ferejohn [2], show with an example that the equilibrium outcome need not be unique. We will show that if there are multiple SSP outcomes they all yield the same SSP payoff vectors. Since the SSP payoff is unique whenever the SSP outcome is unique, we will only consider the case where there are at least two equilibrium outcomes.

Throughout the rest of the paper, let (v, r) and $(\bar{v}, \bar{r})$ be two SSP equilibrium outcomes such that $r \neq \bar{r}$. Let w and $\bar{w}$ denote the corresponding disbursement vectors, and for any player i, let μ_i and $\bar{\mu}_i$ denote the corresponding probabilities of being in the winning coalition when some other player is the proposer.

From (7), we see that, given his recognition probability, the payoff to a player depends only on two factors: the probability that he is in the winning coalition when someone else proposes and his disbursement when he is the proposer. Hence, we start by comparing how the payoff to a player

[5] These monotonicity conditions need not be true in a finite horizon version of the game as the following example due to David Baron illustrates. There are 2 periods, 3 players, a common discount factor δ, recognition probabilities $p_1 = 0.25$, $p_2 = 0.35$, and $p_3 = 0.45$, and the agreement rule is majority rule. In this example, the payoffs for the three players are $0.25 + 0.1\delta$, 0.35, and $0.45 - 0.1\delta$ respectively. For a sufficiently large discount factor, the player with the highest recognition probability gets the lowest payoff.

changes when one or both of these factors change. The results of these comparisons are contained in Lemmata 2–8 and Corollary 3, that are stated and proved in the Appendix.

First note that, if a player is in the winning coalition with a higher probability when he is not the proposer, and if he also pays less to buy votes when lie is the proposer, he cannot be worse off (Lemma 2). Moreover, any favorable change in his disbursement implies an increase in a player's payoff even if his probability of being in the winning coalition decreases (Lemma 6).

The second result may appear counterintuitive by examining the payoff Eq. (7) for one player only. The proof relies on two observations. First, the players can be enumerated as $\{1, ..., n\}$ so that the first $q-1$ players' votes are relatively cheap compared to player q's vote and the last $n-q$ players' votes are relatively expensive compared to player q's vote (Lemma 4). This implies that the probabilities of being in the winning coalition cannot change "much": if a player's vote is strictly cheaper than player q's vote in one equilibrium (so that he is always in the winning coalition when someone else proposes), it cannot be strictly more expensive than player q's vote in another equilibrium (so that he is never in the winning coalition when someone else proposes). Second, a strict decline in the probability of being in the winning coalition implies that another player is in the winning coalition with a strictly higher probability (Lemma 5). Hence, if a player is in the winning coalition with a strictly lower probability, then his payoff must have increased.

Therefore, we conclude that if the disbursement for a player does not change, then his payoff cannot change either (Corollary 3). In other words, in order to establish the uniqueness of SSP payoffs, it is sufficient to show that the disbursement vector does not change (i.e., the disbursement by *each* player is the same in any two equilibria).

Next, note that the disbursements by any two players change in the same direction (Lemma 8). The proof of this result relies on the following two observations. First, the difference between the disbursements of any two players is either zero or is equal to the difference between their discounted payoffs. Second, an increase in the payoff of a player (which, by Lemma 2 and Lemma 6, is possible only when his disbursement decreases) is less than the decrease in his disbursement (Lemma 7). Then, if the disbursements by two players change in the opposite direction, the change cannot be supported with the (smaller) change in the corresponding payoffs. Hence, in order to show the uniqueness of SSP payoffs, it is sufficient to show that the disbursement by any arbitrarily chosen player does not change.

Without loss of generality (see Lemma 4) we assume

$$\delta_i v_i \leqslant \delta_q v_q \qquad \text{and} \qquad \delta_i \bar{v}_i \leqslant \delta_q \bar{v}_q \qquad \text{for all} \quad i \leqslant q,$$

and

$$\delta_i v_i \geqslant \delta_q v_q \qquad \text{and} \qquad \delta_i \bar{v}_i \geqslant \delta_q \bar{v}_q \qquad \text{for all} \quad i \geqslant q.$$

In particular, $w_q = \sum_{i=1}^{q-1} \delta_i v_i$ and $\bar{w}_q = \sum_{i=1}^{q-1} \delta_i \bar{v}_i$.

We can now prove the uniqueness of SSP payoffs.

THEOREM 5. *If* (v, r) *and* $(\bar{v}, \bar{r})$ *are two SSP equilibrium outcomes then* $v = \bar{v}$.

Proof. By Corollary 3 and Lemma 8, it suffices to prove that $\bar{w}_q = w_q$. Suppose not and without loss of generality assume that $\bar{w}_q > w_q$. Then, by Lemma 8, for all $i \in N$, $\bar{w}_i > w_i$. In particular, for $i = 1, ..., q-1$, $\bar{w}_i > w_i$. But by Lemma 7, this implies that $\bar{v}_i < v_i$ for all $i = 1, ..., q-1$, which in turn implies that $\bar{w}_q = \sum_{i}^{q-1} \bar{v}_i < \sum_{i=1}^{q-1} v_i = w_q$. This contradiction proves the desired result. Q.E.D.

APPENDIX

LEMMA 1. *Consider the problems*

$$\min_{x \in S} \quad f(x) \tag{14}$$

and

$$\min_{x \in \bar{S}} \quad f(x), \tag{15}$$

where f is linear, S is finite and $\bar{S}$ *is the convex hull of S. Let X denote the set of minimizers for* (14) *and* $\bar{X}$ *denote the set of minimizers for* (15). *Then* $\bar{X}$ *is the convex hull of X.*

Proof. Let $K(X)$ denote convex hull of X. Note that $K(X) \subset \bar{S}$ and f is linear implies $K(X) \subset \bar{X}$. Thus, it suffices to prove that $\bar{X} \subset K(X)$.

Let $\{x, ..., x_m\}$ be an enumeration of S and for some $k \leqslant m$, without loss of generality, let $\{x_1, ..., x_k\}$ be an enumeration of X. Suppose $\bar{x}$ is a minimizer for (15). Then $\bar{x} \in \bar{X} \subset \bar{S}$. Since $\bar{S}$ is the convex hull of S, there exists $\alpha = (\alpha_1, ..., \alpha_m)$ with the property that $\alpha_i \geqslant 0$ for all i, $\sum_{i=1}^{m} \alpha_i = 1$ and $\bar{x} = \sum_{i=1}^{m} \alpha_i x_i$. Since f is linear, $f(\bar{x}) = \sum_{i=1}^{m} \alpha_i f(x_i)$. Note that $f(x_i) < f(x_j)$ for $i = 1, ..., k$ and $j = k+1, ..., m$. Thus it must be the case that $\alpha_j = 0$ for all $j = k+1, ..., m$ for otherwise $\bar{x}$ does not minimize (15), but then $\bar{x} \in K(X)$. Q.E.D.

All the results below also hold for the equilibrium outcome $(\bar{v}, \bar{r})$ when we replace (w, μ, v, r) with $(\bar{w}, \bar{\mu}, \bar{v}, \bar{r})$.

LEMMA 2. *For all $i \in N$, if $w_i \geqslant \bar{w}_i$ and $\mu_i \leqslant \bar{\mu}_i$, then $v_i \leqslant \bar{v}_i$. The inequality is strict if $w_i \neq \bar{w}_i$ or $\mu_i \neq \bar{\mu}_i$.*

Proof. By (7),

$$v_i = \frac{p_i(1-w_i)}{1-\delta_i \mu_i} \leqslant \frac{p_i(1-w_i)}{1-\delta_i \bar{\mu}_i} \leqslant \frac{p_i(1-\bar{q}_i)}{1-\delta_i \bar{\mu}_i} = \bar{v}_i.$$

The proof follows immediately by noting that at least one of the inequalities in the above expression is strict if $w_i > \bar{w}_i$ or $\mu_i < \bar{\mu}_i$. Q.E.D.

LEMMA 3. *For all $i \in N$, if $\mu_i \geqslant \bar{\mu}_i$, $w_i \geqslant \bar{w}_i$, then $\bar{v}_i - v_i \leqslant w_i - \bar{w}_i$. The inequality is strict if $w_i \neq \bar{w}_i$ or $\mu_i \neq \bar{\mu}_i$.*

Proof. If $v_i > \bar{v}_i$, there is nothing to prove. Suppose $v_i \leqslant \bar{v}_i$. Then, by (7),

$$\begin{aligned} \bar{v}_i - v_i &= \frac{p_i(1-\bar{w}_i)}{1-\delta_i \bar{\mu}_i} - \frac{p_i(1-w_i)}{1-\delta_i \mu_i} \\ &\leqslant \frac{p_i(1-\bar{w}_i)}{1-\delta_i \mu_i} - \frac{p_i(1-w_i)}{1-\delta_i \mu_i} \\ &= \frac{p_i(w_i - \bar{w}_i)}{1-\mu_i \delta_i} \\ &\leqslant w_i - \bar{w}_i, \end{aligned}$$

where the first inequality follows from the fact that $\bar{\mu}_i \leqslant \mu_i$, and the second inequality follows from (6). The result follows immediately since the first inequality is strict if $\bar{\mu}_i < \mu_i$ and the second inequality is strict if $w_i > \bar{w}_i$. Q.E.D.

LEMMA 4. *Let $\{i_1, ..., i_n\}$ be an enumeration of N such that*

$$\delta_{i_1} v_{i_1} \leqslant \cdots \leqslant \delta_{i_n} v_{i_n}$$

and let $\{j_1, ..., j_n\}$ be an enumeration of N such that

$$\delta_{j_1} \bar{v}_{j_1} \leqslant \cdots \leqslant \delta_{j_n} \bar{v}_{j_n}.$$

(i) *Either*

$$\delta_k v_k \leqslant \delta_{i_q} v_{i_q} \Rightarrow \delta_k \bar{v}_k \leqslant \delta_{j_q} \bar{v}_{j_q} \qquad \forall k \in N$$

or

$$\delta_k \bar{v}_k \leqslant \delta_{j_q} \bar{v}_{j_q} \Rightarrow \delta_k v_k \leqslant \delta_{i_q} v_{i_q} \qquad \forall k \in N.$$

(ii) *There exists* $k \in N$ *such that* $\delta_k v_k = \delta_{i_q} v_{i_q}$, *and* $\delta_k \bar{v}_k = \delta_{j_q} \bar{v}_{j_q}$.

Proof. (i) Suppose the assertion is not true. Then, there exist k and k' such that

$$\delta_k v_k \leqslant \delta_{i_q} v_{i_q} \qquad \text{and} \qquad \delta_k \bar{v}_k > \delta_{j_q} \bar{v}_{j_q},$$

and

$$\delta_{k'} \bar{v}_{k'} \leqslant \delta_{j_q} \bar{v}_{j_q} \qquad \text{and} \qquad \delta_{k'} v_{k'} > \delta_{i_q} v_{i_q}.$$

but then

$$\delta_k v_k \leqslant \delta_{i_q} v_{i_q} < \delta_{k'} v_{k'}$$

implies $p_{k'} \delta_{k'} > p_k \delta_k$, and

$$\delta_{k'} \bar{v}_{k'} \leqslant \delta_{j_q} \bar{v}_{j_q} < \delta_k \bar{v}_k$$

implies $p_{k'} \delta_{k'} < p_k \delta_k$ by part (i) of Theorem 4, leading to a contradiction.

(ii) Partition N as.

$$N_1 = \{k \in N : \delta_k v_k < \delta_{i_q} v_{i_q}\},$$
$$N_2 = \{k \in N : \delta_k v_k = \delta_{i_q} v_{i_q}\},$$
$$N_3 = \{k \in N : \delta_k v_k > \delta_{i_q} v_{i_q}\}.$$

Similarly, define $\bar{N}_1$, $\bar{N}_2$ and $\bar{N}_3$ by replacing v with $\bar{v}$ and i_q with j_q.

By part (i) of the Lemma, without loss of generality, we can assume that, for all $k \in N$ $\delta_k \bar{v}_k \leqslant \delta_{j_q} \bar{v}_{j_q}$, whenever $\delta_k v_k \leqslant \delta_{i_q} v_{i_q}$, which implies $N_1 \cup N_2 \subseteq \bar{N}_1 \cup \bar{N}_2$.

Suppose the assertion of part (ii) of the Lemma is not true, i.e., $N_2 \cap \bar{N}_2 = \emptyset$. Then, it must be the case that $N_2 \supseteq \bar{N}_1$. In particular $\delta_{i_q} \bar{v}_{i_q} < \delta_{j_q} \bar{v}_{j_q}$.

Note that $N_1 \cap \bar{N}_2$. If this were not the case, then there would exist a $k \in N$ such that

$$\delta_k v_k < \delta_{i_q} v_{i_q}, \tag{16}$$

and

$$\delta_k \bar{v}_k = \delta_{j_q} \bar{v}_{j_q} > \delta_{i_q} \bar{v}_{i_q}, \tag{17}$$

but (16) implies $\frac{p_k \delta_k}{1-\delta_k} < \frac{p_{i_q} \delta_{i_q}}{1-\delta_{i_q}}$ and (17) implies $\frac{p_k \delta_k}{1-\delta_k} > \frac{p_{i_q} \delta_{i_q}}{1-\delta_{i_q}}$ by part (ii) of Theorem 4. Hence $N_1 \cap \bar{N}_2 = \varnothing$.

Since $(N_1 \cup N_2) \cap \bar{N}_2 = \varnothing$ and $N_1 \cup N_2 \subseteq \bar{N}_1 \cup \bar{N}_2$, it must be the case that $N_1 \cup N_2 \subseteq \bar{N}_1$. But, by definition, $\#(N_1 \cup N_2) \geqslant q$ and $\# \bar{N}_1 \leqslant q-1$, leading to a contradiction. Q.E.D.

LEMMA 5. *Without loss of generality, assume that, for all* $i \leqslant q$, $\delta_i v_i \leqslant \delta_q v_q$ *and* $\delta_i \bar{v}_i \leqslant \delta_q \bar{v}_q$, *and for all* $i \geqslant q$, $\delta_i v_i \geqslant \delta_q v_q$ *and* $\delta_i \bar{v}_i \geqslant \delta_q \bar{v}_q$. *Then, for all* $i \in N$, *if* $\mu_i > \bar{\mu}_i$, *then there exists* $j \in N$ *with* $\mu_j < \bar{\mu}_j$ *such that*

$$\delta_j v_j \geqslant \delta_q v_q \geqslant \delta_i v_i, \tag{18}$$

and

$$\delta_i \bar{v}_i \geqslant \delta_q \bar{v}_q \geqslant \delta_j \bar{v}_j. \tag{19}$$

Furthermore, if any of the inequalities in (18) *is strict, both of the inequalities in* (19) *hold with equality. Similarly, if any of the inequalities in* (19) *is strict, both of the inequalities in* (18) *hold with equality.*

Proof. Let $i \in N$ be given. Since $\mu_i > \bar{\mu}_i$ there must exist at least one player $j \in N$ with $\mu_j < \bar{\mu}_j$, for otherwise at least one player k is not solving (5).

Since $\mu_i > \bar{\mu}_i \geqslant 0$, it must be the case that $\delta_i v_i \leqslant \delta_q v_q$, for otherwise, $\mu_i = 0$ by (9). Also, $1 - p_i \geqslant \mu_i > \bar{\mu}_i$ implies $\delta_i \bar{v}_i \geqslant \delta_q \bar{v}_q$ by (9) again. By a similar argument, we obtain $\delta_j v_j \geqslant \delta_q v_q$ and $\delta_j \bar{v}_j \leqslant \delta_q \bar{v}_q$.

Notice that, if either of the inequalities in (18) is strict, then both of the inequalities of (19) hold with equality. To see, first suppose $\delta_j v_j > \delta_q v_q$. Then, $j > q$ and hence, $\delta_j \bar{v}_j \geqslant \delta_q \bar{v}_q$, which is possible only if the second inequality of (19) is not strict. If also $\delta_i \bar{v}_j > \delta_q \bar{v}_q$, we obtain a contradiction using part (i) of Theorem 4.

Next suppose, $\delta_i v_i < \delta_q v_q$. This implies that $i < q$, and hence, $\delta_i \bar{v}_i \leqslant \delta_q \bar{v}_q$, which is only possible if the first inequality of (19) is not strict. If, in addition, $\delta_j \bar{v}_j < \delta_q \bar{v}_q$, we obtain a contradiction using part (ii) of Theorem 4.

By a similar argument, if any of the inequalities in (19) is strict, both of the inequalities of (18) must hold with equality. Q.E.D.

LEMMA 6. *Without loss of generality, assume that, for all* $i \leqslant q$, $\delta_i v_i \leqslant \delta_q v_q$ *and* $\delta_i \bar{v}_i \leqslant \delta_q \bar{v}_q$, *and for all* $i \geqslant q$, $\delta_i v_i \geqslant \delta_q v_q$, *and* $\delta_i \bar{v}_i \geqslant \delta_q \bar{v}_q$. *Then, for all* $i \in N$, *if* $\mu_i > \bar{\mu}_i$ *and* $w_i \geqslant \bar{w}_i$, *then* $v_i \leqslant \bar{v}_i$. *The inequality is strict if* $w_i \neq \bar{w}_i$.

Proof. Since $\mu_i > \bar{\mu}_i$ by Lemma 5 there exists a player j such that $\mu_j < \bar{\mu}_j$; and (18) and (19) hold. Also, by Lemma 5, there are two possible cases.

Case 1. $\delta_i\bar{v}_i \geqslant \delta_q\bar{v}_q \geqslant \delta_j\bar{v}_j$ and $\delta_j v_j = \delta_q v_q = \delta_i v_i$.

If $v_j < \bar{v}_j$, then $\delta_i v_i = \delta_j v_j < \delta_j\bar{v}_j \leqslant \delta_i\bar{v}_i$ and the result follows. Suppose $v_j \geqslant \bar{v}_j$. Then it must be the case that $w_j \leqslant \bar{w}_j$, for otherwise by Lemma 2 $2v_j < \bar{v}_j$, a contradiction.

By (10),

$$\bar{w}_j - w_j = (\bar{w}_q - w_q) + \delta_q(\bar{v}_q - v_q) - \delta_j(\bar{v}_j - v_j),$$

and hence

$$\delta_q(\bar{v}_q - v_q) = [(\bar{w}_j - w_j) + \delta_j(\bar{v}_j - v_j)] + (w_q - \bar{w}_q).$$

The term in the brackets on the right hand side is nonnegative by Lemma 3 and $(w_q - \bar{w}_q)$ is nonnegative since $w_q = q_i \geqslant \bar{w}_i = \bar{w}_q$. The result follows by noting that $\delta_q v_q = \delta_i v_i$ and $\delta_i\bar{v}_i \geqslant \delta_q\bar{v}_q$.

Case 2. $\delta_i\bar{v}_i = \delta_q\bar{v}_q = \delta_j\bar{v}_j$ and $\delta_j v_j \geqslant \delta_q v_q \geqslant \delta_i v_i$.

If $v_j < \bar{v}_j$, the result follows since $\delta_i v_i \leqslant \delta_j v_j < \delta_j\bar{v}_j = \delta_i\bar{v}_i$. Suppose $v_j \geqslant \bar{v}_j$. Then it must be the case that $w_j \leqslant \bar{w}_j$, for otherwise Lemma 2 implies that $v_j < \bar{v}_j$, a contradiction.

Since $\bar{\mu}_j > \mu_j$, $\bar{w}_j \geqslant w_j$ and $\bar{v}_j < v_j$, by Lemma 3, we obtain

$$0 \leqslant v_j - \bar{v}_j < \bar{w}_j - w_j = \bar{w}_q - w_q.$$

Also, note that by (10), $w_i = w_q + \delta_q v_q - \delta_i v_i$ and $\bar{w}_i = \bar{w}_q = \bar{w}_q + \delta_q\bar{v}_q - \delta_i\bar{v}_i$. Since $w_i \geqslant \bar{w}_i$, we have $\delta_i(\bar{v}_i - v_i) \geqslant (\bar{w}_q - w_q) - \delta_q(v_q - \bar{v}_q)$. The right hand side of this inequality is strictly positive since, $(\bar{w}_q - w_q) > \delta_j(v_j - \bar{v}_j) \geqslant \delta_q(v_q - \bar{v}_q)$. Hence, the result follows. Q.E.D.

COROLLARY 3. *For all* $i \in N$, $v_i = \bar{v}_i$ *if and only if* $w_i = \bar{w}_i$.

LEMMA 7. *Without loss of generality, assume that, for all* $i \leqslant q$, $\delta_i v_i \leqslant \delta_q v_q$, *and* $\delta_i\bar{v}_i \leqslant \delta_q\bar{v}_q$, *and for all* $i \geqslant q$, $\delta_i v_i \geqslant \delta_q v_q$ *and* $\delta_i\bar{v}_i \geqslant \delta_q\bar{v}_q$. *Then, for all* $i \in N$, *if* $w_i \geqslant \bar{w}_i$, *then* $0 \leqslant \bar{v}_i - v_i \leqslant w_i - \bar{w}_i$. *The inequalities are strict if and only if* $w_i > \bar{w}_i$.

Proof. We first prove that the inequalities hold and then show that they are strict only if $w_i > \bar{w}_i$. (That the inequalities are strict if $w_i > \bar{w}_i$ will follow from Lemmata 2, 3 and 6 wherever they are used in the proof.)

That $0 \leqslant \bar{v}_i - v_i$ follows from Lemma 2 and Lemma 6. Thus, it suffices to show that $\bar{v}_i - v_i \leqslant w_i - \bar{w}_i$.

If $\mu_i \geqslant \bar{\mu}_i$, then $\bar{v}_i - v_i \leqslant w_i - \bar{w}_i$ by Lemma 6 and Lemma 3. Suppose $\mu_i < \bar{\mu}_i$. Since $\mu_i < \bar{\mu}_i$, by Lemma 5 there exists a player j such that $\mu_j > \bar{\mu}_j$ and there are two possible cases.

Case 1. $\delta_j \bar{v}_j \geqslant \delta_q \bar{v}_q \geqslant \delta_i \bar{v}_i$ and $\delta_i v_i = \delta_q v_q = \delta_j v_j$.

By (10) $w_j = w_q = w_i$ and $\bar{w}_i \geqslant \bar{w}_q = \bar{w}_j$. Then, $w_j \geqslant \bar{w}_j$. Since also $\mu_j > \bar{\mu}_j$, by Lemma 6 and Lemma 3, $0 \leqslant \bar{v}_j - v_j \leqslant w_j - \bar{w}_j$. Notice that $w_j - \bar{w}_j = w_q - \bar{w}_q$ and $\bar{v}_j - v_j \geqslant \delta_j(\bar{v}_j - v_j) \geqslant \delta_q(\bar{v}_q - v_q)$. Hence, $w_q - \bar{w}_q \geqslant \delta_q(\bar{v}_q - v_q)$. Also, by (10)

$$w_i - \bar{w}_i = [(w_q - \bar{w}_q) - \delta_q(\bar{v}_q - v_q)] + \delta_i(\bar{v}_i - v_i).$$

The result follows since the term in the brackets is nonnegative.

Case 2. $\delta_j \bar{v}_j = \delta_q \bar{v}_q = \delta_i \bar{v}_i$ and $\delta_i v_i \geqslant \delta_q v_q \geqslant \delta_j v_j$.

By (10) $w_j \geqslant w_q = w_i$ and $\bar{w}_i = \bar{w}_j = \bar{w}_q$. Then, $w_j \geqslant \bar{w}_j$. Since also $\mu_j > \bar{\mu}_j$ by Lemma 6 and Lemma 3, $0 \leqslant \bar{v}_j - v_j \leqslant w_j - \bar{w}_j$.

Using (10) again, we obtain $w_j = w_q + \delta_q v_q - \delta_j v_j \leqslant w_i + \delta_i v_i - \delta_j v_i$ and $\bar{w}_j = \bar{w}_q + \delta_q \bar{v}_q - \delta_j \bar{v}_j = \bar{w}_i + \delta_i \bar{v}_i - \delta_j \bar{v}_j$. Hence,

$$w_i - \bar{w}_i \geqslant [(w_j - \bar{w}_j) - \delta_j(\bar{v}_j - v_j)] + \delta_i(\bar{v}_i - v_i).$$

The result follows by noting that the term in the brackets is nonnegative.

To see that the inequalities are strict only if $w_i > \bar{w}_i$, notice that, if $w_i = \bar{w}_i$, then $0 \leqslant \bar{v}_i - v_i \leqslant w_i - \bar{w}_i = 0$ which in turn implies that neither inequality can be strict. Q.E.D.

LEMMA 8. *Without loss of generality, assume that, for all* $i \leqslant q$, $\delta_i v_i \leqslant \delta_q v_q$, *and* $\delta_i \bar{v}_i \leqslant \delta_q \bar{v}_q$, *and for all* $i \geqslant q$, $\delta_i v_i \geqslant \delta_q v_q$ *and* $\delta_i \bar{v}_i \geqslant \delta_q \bar{v}_q$. *Then, for all* $i \in N$, $w_i \geqslant \bar{w}_i$ *if* $w_q \geqslant \bar{w}_q$. *The inequality is strict if and only if* $w_q > \bar{w}_q$.

Proof. If $i \geqslant q$, the proof follows immediately by (10), since $\delta_i v_i \geqslant \delta_q v_q$, and $\delta_i \bar{v}_i \geqslant \delta_q \bar{v}_q$. So, let $i < q$. Then, $\delta_i v_i \leqslant \delta_q v_q$, and $\delta_i \bar{v}_i \leqslant \delta_q \bar{v}_q$ and hence, from (10)

$$w_i - \bar{w}_i = [(w_q - \bar{w}_q) - \delta_q(\bar{v}_q - v_q)] + \delta_i(\bar{v} - v_i) \geqslant \delta_i(\bar{v}_i - v_i), \qquad (20)$$

where the inequality follows from Lemma 7 and is strict if and only if $w_q > \bar{w}_q$.

Now if $w_i < \bar{w}_i$, then, by Lemma 7, $0 < v_i - \bar{v}_i < \bar{w}_i - w_i$ and by (20) $\bar{w}_i - w_i \leqslant \delta_i(v_i - \bar{v}_i) \leqslant v_i - \bar{v}_i$, a contradiction. Hence, $w_i \geqslant \bar{w}_i$ and the inequality is strict if $w_q > \bar{w}_q$ by (20). Finally, if $w_q = \bar{w}_q$, then $\bar{v}_q = v_q$ and hence, by (20), $w_i - \bar{w}_i = \delta_i(\bar{v}_i - v_i) \geqslant \bar{v}_i - v_i$, which is possible only if $w_i = \bar{w}_i$. Q.E.D.

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